Introduction to Wind Energy

By E.H. Lysen

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Introduction to wind energy

Basic and advanced introduction to wind energy
with emphasis on water pumping windmills

By:
E.H. Lysen

May 1983
ACKNOWLEDGEMENT

Most of the material contained in this publication is based upon the work of the Wind Energy Group of the Eindhoven University of Technology, the Netherlands, guided by Paul T. Smulders. Part of the work has been carried out by graduate students in the University Wind Energy programme and a large part stems from the work of the staff employed by the programme for the Steering Committee Wind Energy Developing Countries (SWD), funded by the Netherlands Minister of Development Co-operation. The latter programme focuses on the application of wind energy for water pumping.

I have brought together the work of the following persons, mentioned in alphabetical order:

Jos Beurskens: aerodynamic theory
Peter van de Does: dynamic behaviour of piston pumps
Kees Heil: aerodynamic theory
Dirk Hengeveld: matching generators and rotors
Geert Hoapers: air chambers, piston pumps, hinged vane safety system
Martin Houet: aerodynamic theory
Wim Jansen: rotor design
Gerard de Leede: starting behaviour rotor + pump
Erik Lysen: output prediction, coupling pumps and rotors, economics
Joop van Meel: pumps with leakhole, starting behaviour rotor, wind statistics
Adrie Kragten: piston pumps, hinged vane safety system, rotor design
Leo Paulissen: analysis wind regimes, matching generators and rotors
Rein Schermerhorn: forces on rotors
Paul Smulders: aerodynamic theory, rotor design, analysis wind regimes, output prediction
Jan Snoeij: valve behaviour in piston pumps
A large number of students, not mentioned here, have made contributions to many of the subjects discussed.

Special thanks I am indebted to Ms. Ratrie and Ms. Varin of the Energy Technology Division, Asian Institute of Technology, Bangkok, who typed the first draft of this Introduction, and to Mrs. Riet Bedet, of the Wind Energy Group in Eindhoven, who has spent tiring hours before the screen, new for her, to get text and formulas on the floppy disk. Thanks also to Ms. Ruth Gruyters who has professionally redrawn my sketches and to Mr. Jan van Haaren for screening my English.

E.L.
Eindhoven
January 1982

The second edition

The second edition is identical to the first edition, except that some misleading definitions have been revised and typing errors have been corrected. Suggestions for further improvements are most welcome.

E.L.
Amersfoort
May 1983
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\begin{align*}
S & \text{ velocity duration function} \\
t & \text{time} \\
T & \text{length of period} \\
T & \text{dimensionless time (section 5.5.5)} \\
T & \text{thrust force} \\
T & \text{temperature} \\
T & \text{function (section 11.2)} \\
u & \text{component of wind velocity in } x\text{-direction} \\
U & \text{speed} \\
v & \text{component of wind velocity in } y\text{-direction} \\
V & \text{wind velocity} \\
V_1, V_m & \text{undisturbed wind velocity} \\
\bar{V} & \text{average wind velocity} \\
V_2 & \text{wind velocity far behind the rotor} \\
w & \text{component of wind velocity in } z\text{-direction} \\
W & \text{relative wind velocity} \\
W & \text{work} \\
x & \text{coordinate} \\
x & \text{relative radius (r/R)} \\
x & \text{reduced wind velocity } (V/\bar{V}) \\
y & \text{coordinate} \\
z & \text{coordinate height} \\
z_0 & \text{roughness height}
\end{align*}
\( \alpha \) angle of attack profile (or vane)  
\( \beta \) blade setting angle  
\( \beta \) damping coefficient (section 5.5.4)  
\( \gamma \) angle between actual position of main vane and it rest position  
\( \gamma \) exponent in gas law  
\( \Gamma \) circulation  
\( \Gamma(x) \) gamma function  
\( \delta \) angle of yaw (rotor axis - wind direction)  
\( \epsilon \) angle between hinge axis and vertical axis of rotor head (in plane of vertical axis and rotor axis)  
\( \eta \) efficiency  
\( \Theta \) blade position angle  
\( \nu \) volume  
\( \lambda \) tip speed ratio  
\( \lambda \) local speed ratio  
\( \mu \) dynamic viscosity  
\( \mu \) constant (section 5.4)  
\( \nu \) kinematic viscosity  
\( \xi \) angle between auxiliary vane and rotor plane  
\( \pi \) 3.14159265359  
\( \rho \) density  
\( \sigma \) standard deviation  
\( \sigma \) tensile stress (ch. 10)  
\( \sigma \) solidity ratio of a rotor  
\( \tau \) time constant  
\( \tau \) shearing stress (ch. 10)  
\( \tau \) tilting angle of rotor shaft  
\( \tau \) availability (section 9.2.6)  
\( \phi \) dimensionless flow (section 5.5.5)  
\( \phi \) angle between relative wind direction (W) and rotor plane (\( \phi = \alpha + \beta \))  
\( \psi \) phase angle (section 5.5.4)  
\( \omega \) induced tangential angular wind velocity  
\( \Omega \) angular velocity of rotor
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* The combined indices of chapter 10 are explained at the beginning of chapter 10.
1. INTRODUCTION

This publication is the written result of three courses given at the Asian Institute of Technology in Bangkok, one in 1980 and two in 1981, funded by the Netherlands Ministry of Development Co-operation.

It is based upon the information and experience contained in SWD publications, in the internal publications of the SWD participants (see the Acknowledgement), and in the open literature. Some subjects have never been published before, such as the behaviour of the hinged vane safety system, the dynamic behaviour of valves in piston pumps and the starting behaviour of windmills with piston pumps.

The texts have been written when the author was a member of the Wind Energy Group of the Eindhoven University of Technology, the Netherlands.

The reader is assumed to have a knowledge of physics and mathematics on an undergraduate level. The publication aims at practical applications such as analyzing wind regimes, designing wind rotors, calculating energy outputs etc., but also provides the mathematics behind these. Construction details of windmills will hardly be touched upon, however, as they are the subject of special SWD publications. Also a general introduction on the history of windmills and their different types and applications is omitted, because they can be found in many excellent books [1-4].

Nearly each chapter of this publication is divided into two parts, an introductory part and an advanced part, as indicated in the Contents.

The introduction parts together form a complete introductory course, which could be given separately. These parts at the same time prepare for the understanding of the advanced parts, presented in eight of the twelve chapters.
2. AVAILABLE POWER AND SITE SELECTION

2.1. Available wind power

Air mass flowing with a velocity $V$ through an area $A$ represents a mass flow rate $\dot{m}$ of:

$$\dot{m} = \rho A V \quad (kg/s) \quad (2.1)$$

and thus a flow of kinetic energy per second or kinetic power $P_{kin}$ of:

$$P_{kin} = \frac{1}{2} (\rho A V)^2 = \frac{1}{2} \rho A V^3 \quad (W) \quad (2.2)$$

where

- $\rho = \text{air density} \ (kg/m^3)$
- $A = \text{area swept by the rotor blades} \ (m^2)$
- $V = \text{undisturbed wind velocity} \ (m/s)$

Fig. 2.1 A volume $V*A$ of air is flowing every second through an area $A$. This represents a mass flow rate of $\rho A V \ (kg/s)$. 
In words this relation expresses three things:

1. Wind power is proportional to the density of the air. This means that high in the mountains one gets less power at the same wind speed. A table of air densities for different temperatures and heights is given in Appendix A.

2. Wind power is proportional to the area swept by the rotor blades, or is proportional to the square of the diameter of the rotor.

3. Wind power is proportional to the cube of the wind velocity, so it pays to carefully select a good site for a windmill: 10% more wind gives 30% more power.

A wind rotor can only extract power from the wind, because it slows down the wind: the wind speed behind the rotor is lower than in front of the rotor. Too much slowing down causes the air to flow around the wind rotor area instead of through the area and it turns out [5] that the maximum power extraction is reached when the wind velocity in the wake of the rotor is 1/3 of the undisturbed wind velocity $V_\infty$. In that case the rotor itself "feels" a velocity $2/3 V_\infty$, so the effective mass flow is only $\rho A (2/3 V_\infty)$. If this mass flow is slowed down from $V_\infty$ to $1/3 V_\infty$ the extracted power is equal to:

$$ P_{\text{max}} = \frac{1}{2} (\rho A \frac{2}{3} V_\infty)^2 V_\infty - \frac{1}{2} (\rho A \frac{2}{3} V_\infty) (\frac{1}{3} V_\infty)^2 \quad (2.3) $$

or

$$ P_{\text{max}} = \frac{16}{27} \frac{1}{2} \rho A V_\infty^3 $$
In other words, theoretical maximum fraction of extracted power is \( \frac{16}{27} \) or 59.3\%.

This maximum is called the Betz-maximum in honour of the wind pioneer who first derived its value.

The fraction of extracted power, which we call power coefficient \( C_p \), in practice seldom exceeds 40\% if measured as the mechanical power of a real wind rotor. The subsequent conversion into electrical power or pumping power gives a reduction in available power, depending on the efficiency \( \eta \) of transmission and pump or generator. A further reduction of the available power is caused by the fluctuations in speed and direction which an actual windmill experiences in the field.

For a waterpumping windmill these effects lead to the next rule of thumb for a first estimate of the average hydraulic output at a site with an average windspeed \( V \):

\[
\overline{P}_{\text{hydr}} = 0.1 A V^3 \quad (W)
\]  

(2.4)

For electricity generating wind turbines this factor 0.1 must be increased to 0.2 or sometimes 0.25 with good wind turbines.

The flow of water pumped over a head of \( H \) meter by the hydraulic power \( P_{\text{hydr}} \) is given by:

\[
q = \frac{P_{\text{hydr}}}{\rho g H} \quad m^3/s
\]

where: \( \rho = 1000 \text{ kg/m}^3 \)
\( g = 9.8 \text{ m/s}^2 \)

This expression can be reduced to

\[
q = \frac{P_{\text{hydr}}}{g H} \quad \text{liter/s}
\]
As an example we estimate the average output of a ø 5 m windmill at a site with \( V = 3 \) m/s, at a head of 5 m:

\[
q = \frac{0.1 \times \frac{\pi}{4} \times 5^2 \times 3^3}{9.8 \times 5} = 1.1 \text{ liter/s}
\]

In fig. 2.2 the water output can be read for different rotor diameters and heads.

Note: It must be emphasized that these values are only first-guess estimates. As soon as more data about windmill and wind regime are available better estimates can be made as we will see in Chapter 9.

2.2 Site selection

The power output of a wind rotor increases with the cube of the wind speed, as we have seen in section 2.1. This means that the site for a windmill must be chosen very carefully to ensure that the location with the highest wind speed in the area is selected. The site selection is rather easy in flat terrain but much more complicated in hilly or mountainous terrains.

A number of effects have to be considered [6]:

1. windshear: the wind slows down, near the ground, to an extent determined by the surface roughness.
2. turbulence: behind building, trees, ridges etc....
3. acceleration: (or retardation) on the top of hills, ridges etc.
Fig. 2.2 Chart to estimate the output of a water pumping windmill with a given diameter and a given water lifting head, operating in a wind regime with annual (or monthly) wind speed $\bar{V}$. The chart is based upon the following output estimation: $P/A = 0.1\pi\bar{V}^3 \text{ W/m}^2$. 
2.2.1 Windshear

Vegetation, buildings and the ground itself cause the wind to slow down near the ground or, vice versa, the wind speed increases with increasing height. The rate of increase with height strongly depends upon the roughness of the terrain and the changes in this roughness. For various types of terrain the "roughness height" $z_o$ can be determined, usually by means of a gust analysis [7].

flat : beach, ice, snow landscape, ocean $z_o = 0.005$ m
open : low grass, airports, empty crop land $z_o = 0.03$ m
: high grass, low crops $z_o = 0.10$ m
rough : tall row crops, low woods $z_o = 0.25$ m
very rough: forests, orchards $z_o = 0.50$ m
closed : villages, suburbs $z_o = 1.0$ m
towns : town centres, open spaces in forests $z_o > 2$ m

These values can be used in the standard formula for the logarithmic profile of the windshear:

$$\frac{V(z)}{V(z_r)} = \frac{\ln (z/z_o)}{\ln (z_r/z_o)} \quad (2.5)$$

For a reference height of $z_r = 10$ m this formula is shown in fig. 2.3. for different values of the roughness height $z_o$. The graph can be used in areas where there are no large hills or other large obstructions within a range of 1 to 2 km from the windmill.

Note: Formula (2.5) gives the windshear in one location. In case one wants to compare two locations, each with its own roughness height then Wieringa's assumption [7] that the wind speed at 60 m height is unaffected by the roughness, leads to the formula:
Fig. 2.3 The windshear related to a reference height of 10 m, for various roughness heights $z_0$. 

\[ \frac{V(z)}{V(10)} = \frac{\ln(z/z_0)}{\ln(10/z_0)} \]
\( \frac{V(z)}{V(z_r)} = \frac{\ln \left( \frac{60/z}{z_o} \right) \ln \left( \frac{z/z_o}{z_r/z_o} \right)}{\ln \left( \frac{60/z}{z_o} \right) \ln \left( \frac{z/z_r}{z_o/z_r} \right)} \) \hspace{1cm} (2.6)

with \( z_o \) the roughness height at the reference location, for example a meteorological station, where the wind speed is being measured at a reference height \( z_r \).

2.2.2 Turbulence

Wind flowing around buildings or over very rough surfaces exhibits rapid changes in speed and/or direction, called turbulence. This turbulence decreases the power output of the windmill and can also lead to unwanted vibrations of the machine.

In fig. 2.4 the region of turbulence behind a small building is shown.

Fig. 2.4 Zone of turbulence over a small building (after [6]).
The same situation applies near shelterbelts of trees: the turbulence is felt up to a leeward distance of at least 10-15 times the height of the trees. The region of turbulence also extends windward about five times the height of the obstruction [6].

A simple method to detect turbulence and the height to which it extends, is by means of a 1 m. long ribbon tied to a long pole or a kite. The flapping of the ribbon indicates the amount of turbulence.

2.2.3 Acceleration on ridges

Apart from the fact that tops of ridges experience higher wind speeds due to the effect of windshear (2.2.1), the ridge also acts as a sort of concentrator for the air stream, causing the air to accelerate nearby the top (figure 2.5).

Fig. 2.5 Acceleration of the wind over a ridge.
Generally, it can be said that the effect is stronger when the ridge is rather smooth and not too steep nor too flat. The ideal slope angle is said to be 16° (29 m rise per 100 m horizontal distance) but angles between 6° and 16° are good [6]. Angles greater than 27° should be avoided. Triangular shaped ridges are even better than rounded ridges.

The orientation of the ridge should preferably be perpendicular to the prevailing wind direction. If the ridge is curved it is best if the wind blows in the concave side of the ridge.

A quantitative indication of the acceleration is difficult to give, but increases of 10% to 20% in wind speed are easily attained. Isolated hills give less acceleration than ridges, because the air tends to flow around the hill. This means that in some cases the two hill sides, perpendicular to the prevailing wind, are better locations than the top.

For specific cases, such as passes and saddles, valleys and hills, the reader is referred to the Siting Handbook [6] and to Goldings' book [2].
3. ANALYSIS OF WIND REGIMES

3.1. General

In this chapter a number of manipulations with wind data are described. These manipulations are meant to facilitate the judgement to what extent a given location might be suitable for the utilization of wind energy. In this respect we are interested in the answers to questions such as:

- What is the daily, monthly, annual wind pattern?
- What is the duration of low wind speeds, high wind speeds?
- Which wind speeds can we expect at locations not too far from the place of measurement?
- What is the maximum gust speed?
- How much energy can be produced per month, per year?

The question about the energy extraction has been briefly discussed in chapter 2 and will be further explored in chapter 9. Here we will discuss the wind pattern as such and its characterization by numbers and graphs. We assume that a set of hourly data from a meteorological station is available, possibly supplemented by shorttime measurements at the location where a new windmill is planned. When only monthly average wind speeds are available, the rules of thumb in chapter 2 can be used. If no data are available at all, then judgement is limited to enquiries of local population and analysis of the vegetation. The latter is treated rather extensively in the Siting Handbook [6].

The reliability of the data is not questioned here, but in a practical situation it is absolutely necessary to check the actual position of the anemometer, the distance and height of nearby buildings, the type and quality of the anemometer, the method of reading or recording the data, the handling of powercuts and, last but not least, to make sure whether m/s, knots, miles per hours or other units of measurement are employed.
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The hourly wind speeds, which form the basis of our analysis, can be determined in several ways:

- the run-of-wind average of the full hour
- the average of a graph of the full hour
- the average of a graph during the last 10 minutes of each hour (WMO Standard)
- the average of several "snapshot" measurements within the hour.

An example of one month of data recorded in Praia (Cabo Verde) is shown in Fig. 3.1 and it is assumed that for a good reference station such data are available for a number of years. We shall now describe a number of manipulations with these data, basically looking at two aspects:

- time distribution
- frequency distribution

The maximum gust speed cannot be found from the hourly averages but should have been recorded separately.

NOTE: although we are only considering the manipulation with existing data, it is often interesting to know the predictive value of the data measured. Generally it can be stated that the annual wind regime repeats itself quite consistently. The work of Corotis [8], Justus [9] and Ramsdell et al. [10] indicates that the annual average wind speed as found from 12 months of data recording will be within 10% of the true long term average wind speed with a 90% confidence level. See also the work of Cherry [11], who states that the wind is generally more consistent at sites with higher mean wind speeds.
3.2 Time distribution

Plotting the monthly averages of each hour of the day shows the diurnal fluctuations of the wind speed in that particular month (fig. 3.2); in the same figure also the monthly average is shown.

Fig. 3.2 Diurnal pattern of the wind speed at Praia airport in the month of June 1975.

In a similar manner the monthly averages can be plotted to show the monthly fluctuations of the wind speed, compared with the annual average wind speed (fig. 3.3).

A third type of information which can be extracted from the time distribution of the data is the distribution of periods with low wind speeds (lulls). In other words: how often did it happen that the wind speed was lower than, for example, 2 m/s during 12 hours or during two days? This type of information is valuable for the calculation of the size of storage tanks.
Fig. 3.3 The monthly average wind speeds at Praia airport in the year 1978.

The procedure is rather time-consuming, when done by hand, and consists of the following steps:

- choose a reference wind speed $V_{ref}$
- look for the first hour with $V < V_{ref}$ and start counting hours until an hour with $V > V_{ref}$ is encountered
- this means that the first period with $V < V_{ref}$ is found, of which the length should be noted by adding one unit to the period corresponding with that length
- continue until the end of the month (year)
- choose a new reference wind speed
- etc.

The result of this procedure for the data of fig. 3.1 is shown in fig. 3.4
Fig. 3.4 Distribution of the number of periods that the wind speed was smaller than a given value during a consecutive number of hours (data of Praia airport, June 1975).

<table>
<thead>
<tr>
<th>hours</th>
<th>V&lt;2m/s</th>
<th>&lt;3m/s</th>
<th>&lt;4m/s</th>
<th>&lt;5m/s</th>
<th>&lt;6m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3.3 Frequency distribution

Apart from the distribution of the wind speeds over a day or a year it is important to know the number of hours per month or per year during which the given wind speeds occurred, i.e. the frequency distribution of the wind speeds. To arrive at this frequency distribution we must first divide the wind speed domain into a number of intervals, mostly of equal width of 1 m/s or 0.5 m/s. Then, starting at the first interval of say 0-1 m/s, the number of hours is counted in the period concerned that the wind speed was in this interval. When the number of hours in each interval is plotted against the wind speed, the frequency distribution emerges as a histogram (fig. 3.5, from data of fig. 3.1).

<table>
<thead>
<tr>
<th>Interval (m/s)</th>
<th>Hours/ Month</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>0</td>
</tr>
<tr>
<td>1-2</td>
<td>6</td>
</tr>
<tr>
<td>2-3</td>
<td>13</td>
</tr>
<tr>
<td>3-4</td>
<td>32</td>
</tr>
<tr>
<td>4-5</td>
<td>70</td>
</tr>
<tr>
<td>5-6</td>
<td>120</td>
</tr>
<tr>
<td>6-7</td>
<td>126</td>
</tr>
<tr>
<td>7-8</td>
<td>56</td>
</tr>
<tr>
<td>8-9</td>
<td>89</td>
</tr>
<tr>
<td>9-10</td>
<td>81</td>
</tr>
<tr>
<td>10-11</td>
<td>64</td>
</tr>
<tr>
<td>11-12</td>
<td>42</td>
</tr>
<tr>
<td>12-13</td>
<td>11</td>
</tr>
<tr>
<td>13-14</td>
<td>9</td>
</tr>
<tr>
<td>14-15</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>720</td>
</tr>
</tbody>
</table>

Fig. 3.5 The velocity frequency data for Praia airport (June 1975), both in a table and in a histogram.
The top of this histogram, being the most frequent wind speed, is generally not the average wind speed. In trade wind areas with quite steady wind speeds this might be the case, but in other climates the average wind speed is generally higher than the most frequent wind speed (see also fig. 3.10). The average wind speed of a given frequency distribution is calculated as follows:

\[
\bar{V} = \frac{t_1 V_1 + t_2 V_2 + \ldots + t_i V_i + \ldots + t_n V_n}{t_1 + t_2 + \ldots + t_n}
\]  \hspace{1cm} (3.1)

where: 
- \(t_i\) : number of hours in windspeed interval \(i\)
- \(V_i\) : middle of wind speed interval \(i\)
- \(V\) : average wind speed

The thus calculated average wind speed obviously must be equal to the average wind speed as calculated from the original data by taking the sum of all hourly data and dividing them by their number.

The frequency distribution will be used to calculate the energy output of a windmill by multiplying the number of hours in each interval with the power output that the windmill supplies at that wind speed interval (chapter 9).

It is often important to know the number of hours that a windmill will run or the time fraction that a windmill produces more than a given power. In this case it is necessary to add the number of hours in all intervals above the given wind speed. The result is the duration distribution which is easily found by adding the number of hours of each interval to the sum of all hours of the higher intervals. So it is best to start with the highest interval, with zero hours of wind speed higher than the upper boundary of the interval and subsequently add the number of hours of the next lower interval, etc. This is done in fig. 3.6 with data of fig. 3.1.
<table>
<thead>
<tr>
<th>interval</th>
<th>frequency</th>
<th>duration V &gt; V'</th>
<th>cumulative V &lt; V'</th>
</tr>
</thead>
<tbody>
<tr>
<td>m/s</td>
<td>hours</td>
<td>hours</td>
<td>hours</td>
</tr>
<tr>
<td>0-1</td>
<td>0</td>
<td>720</td>
<td>0</td>
</tr>
<tr>
<td>1-2</td>
<td>6</td>
<td>714</td>
<td>6</td>
</tr>
<tr>
<td>2-3</td>
<td>13</td>
<td>701</td>
<td>19</td>
</tr>
<tr>
<td>3-4</td>
<td>32</td>
<td>669</td>
<td>51</td>
</tr>
<tr>
<td>4-5</td>
<td>70</td>
<td>599</td>
<td>121</td>
</tr>
<tr>
<td>5-6</td>
<td>120</td>
<td>479</td>
<td>241</td>
</tr>
<tr>
<td>6-7</td>
<td>126</td>
<td>353</td>
<td>367</td>
</tr>
<tr>
<td>7-8</td>
<td>56</td>
<td>297</td>
<td>423</td>
</tr>
<tr>
<td>8-9</td>
<td>89</td>
<td>208</td>
<td>512</td>
</tr>
<tr>
<td>9-10</td>
<td>81</td>
<td>127</td>
<td>593</td>
</tr>
<tr>
<td>10-11</td>
<td>64</td>
<td>63</td>
<td>657</td>
</tr>
<tr>
<td>11-12</td>
<td>42</td>
<td>21</td>
<td>699</td>
</tr>
<tr>
<td>12-13</td>
<td>11</td>
<td>10</td>
<td>710</td>
</tr>
<tr>
<td>13-14</td>
<td>9</td>
<td>1</td>
<td>719</td>
</tr>
<tr>
<td>14-15</td>
<td>1</td>
<td>0</td>
<td>720</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>720</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3.6 The velocity frequency data of Praia (June 1975) are transformed in a duration distribution and a cumulative distribution. The upper boundary of the interval is indicated by V'.

The duration values are commonly plotted with the wind speed on the y-axis, as shown in fig. 3.7(a). Here the length of each horizontal column indicates the duration of the time that the wind speed was higher than the upper boundary of the wind speed interval. If the histogram is approximated by a smooth curve through the values at the middle of each interval then a duration curve results. By
studying the shape of this duration curve an idea is obtained about the kind of wind regime. The flatter the duration curve, i.e. the longer one specific wind speed persists, the more constant the wind regime is. The steeper the duration curve, the more irregular the wind regime is. These characteristics will be analyzed mathematically in section 3.4.

In some cases it is preferred to plot the time during which the wind speed was smaller than a given wind speed, and when this is plotted versus the wind speed a cumulative distribution results (fig. 3.7(b)).

Fig. 3.7 Histograms of the duration distribution (a) and the cumulative distribution (b) for Praia (June 1975) as presented in fig. 3.6.
3.4 **Mathematical representation of wind regimes**

3.4.1 General

After having drawn a number of velocity duration histograms or velocity frequency histograms and approximating them by smooth curves it is striking to notice that the shape of these curves is quite similar. This is even clearer if the wind speed values are made dimensionless by dividing them by the average wind speed of that particular distribution. It is quite logical in such a situation to look for mathematical functions that approach the frequency and duration curves as closely as possible, as a tool to predict the output of windmills later on.

In this respect much attention has been given to the Weibull function, since it is a good match with the experimental data [12-15]. In some cases the Rayleigh distribution, a special case of the Weibull distribution, is preferred. This section deals with the Weibull function and the method to estimate its parameters from a given distribution.

Two functions will be used throughout this section \((V > 0)\):

1. the cumulative distribution function \(F(V)\), indicating the time fraction or probability that the wind speed \(V\) is smaller than or equal to a given wind speed \(V'\):

\[
F(V) = P (V < V') \quad \text{(dimensionless)} \quad (3.2)
\]

2. the probability density function, represented in our case by the velocity frequency curve:

* This section is largely based upon the work on Weibull distributions by Stevens and Smulders [12, 13].
\[ f(V) = \frac{d F(V)}{d V} \quad (s/m) \]  

or

\[ F(V) = \int_0^V f(V') d V' \]

The velocity duration function \( S(V) \), defined as the time fraction or probability that the wind speed \( V \) is larger than a given wind speed \( V' \) can be written as:

\[ S(V) = 1 - F(V) = P(V > V') \quad \text{dimensionless} \quad (3.4) \]

The average wind speed \( \bar{V} \) can be found with

\[ \bar{V} = \int_0^\infty V f(V) d V \quad (m/s) \quad (3.5) \]

and the variance is given by

\[ \sigma^2 = \int_0^\infty (V - \bar{V})^2 f(V) d V \quad (m^2/s^2) \quad (3.6) \]

where \( \sigma \) is the standard deviation.

### 3.4.2 The Weibull distribution

The Weibull distribution is characterized by two parameters: the shape parameter \( k \) (dimensionless) and the scale parameter \( c \) (m/s).

The cumulative distribution function is given by:

\[ F(V) = 1 - \exp \left[ - \left( \frac{V}{c} \right)^k \right] \quad (3.7) \]

and the probability density function by
\[ f(V) = \frac{dF}{dV} = \frac{k}{c} \left( \frac{V}{c} \right)^{k-1} \exp \left[ -\left( \frac{V}{c} \right)^k \right] \] (3.8)

With expression (3.5) the average wind speed can be expressed as a function of \( c \) and \( k \) or, vice versa, \( c \) is a function of \( V \) and \( k \). The integral found cannot be solved however, but it can be reduced to a standard integral, the so-called gamma function:

\[ \Gamma(x) = \int_0^\infty y^{x-1} e^{-y} \, dy \] (3.9)

with \( y = \left( \frac{V}{c} \right)^k \) and \( \frac{V}{c} = y^{x-1} \) one obtains \( x = 1 + \frac{1}{k} \) and after a few manipulations:

\[ \bar{V} = c \Gamma \left( 1 + \frac{1}{k} \right) \] (3.10)

Introducing (3.10) in the expressions for \( F(V) \) and \( f(V) \) yields:

\[ F(V) = 1 - \exp \left[ -\Gamma \left( 1 + \frac{1}{k} \right) \left( \frac{V}{c} \right)^k \right] \] (3.11)

and

\[ f(V) = \frac{k}{\bar{V}} \left( \frac{V}{\bar{V}} \right)^k \Gamma \left( 1 + \frac{1}{k} \right) \exp \left[ -\Gamma \left( 1 + \frac{1}{k} \right) \left( \frac{V}{c} \right)^k \right] \] (3.12)

As mentioned in 3.4.1 the Rayleigh distribution is a special case of the Weibull distribution, this for \( k = 2 \). In this case the above expressions reduce to rather simple expressions, noting that:

for \( k = 2 \): \( \Gamma^2 \left( 1 + \frac{1}{k} \right) = \frac{\pi^2}{4} \)

The expressions for \( F(V) \) and \( f(V) \) now become

\[ F(V) = 1 - \exp \left[ -\frac{\pi}{4} \left( \frac{V}{\bar{V}} \right)^2 \right] \] (3.13)

and

\[ f(V) = \frac{\pi}{2} \frac{V}{\bar{V}^2} \exp \left[ -\frac{\pi}{4} \left( \frac{V}{\bar{V}} \right)^2 \right] \] (3.14)
For other values of $k$ the next table applies:

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\Gamma(1 + \frac{1}{k}) = \frac{V}{c}$</th>
<th>$\Gamma^k (1 + \frac{1}{k})$</th>
<th>$G$</th>
<th>$G/\Gamma^k (1 + \frac{1}{k})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.002000</td>
<td>100.2 %</td>
</tr>
<tr>
<td>1.25</td>
<td>0.931384</td>
<td>0.914978</td>
<td>0.915200</td>
<td>100.024%</td>
</tr>
<tr>
<td>1.5</td>
<td>0.902745</td>
<td>0.857724</td>
<td>0.857333</td>
<td>99.954%</td>
</tr>
<tr>
<td>1.6</td>
<td>0.896574</td>
<td>0.839727</td>
<td>0.839250</td>
<td>99.943%</td>
</tr>
<tr>
<td>1.7</td>
<td>0.892244</td>
<td>0.823802</td>
<td>0.823294</td>
<td>99.938%</td>
</tr>
<tr>
<td>1.8</td>
<td>0.889287</td>
<td>0.809609</td>
<td>0.809111</td>
<td>99.938%</td>
</tr>
<tr>
<td>1.9</td>
<td>0.887363</td>
<td>0.796880</td>
<td>0.796421</td>
<td>99.942%</td>
</tr>
<tr>
<td>2.0</td>
<td>0.886227</td>
<td>0.785398($\frac{\pi}{4}$)</td>
<td>0.785000</td>
<td>99.949%</td>
</tr>
<tr>
<td>2.1</td>
<td>0.885694</td>
<td>0.774989</td>
<td>0.774667</td>
<td>99.958%</td>
</tr>
<tr>
<td>2.2</td>
<td>0.885625</td>
<td>0.765507</td>
<td>0.765273</td>
<td>99.969%</td>
</tr>
<tr>
<td>2.3</td>
<td>0.885915</td>
<td>0.756835</td>
<td>0.756696</td>
<td>99.981%</td>
</tr>
<tr>
<td>2.4</td>
<td>0.886482</td>
<td>0.748873</td>
<td>0.748833</td>
<td>99.995%</td>
</tr>
<tr>
<td>2.5</td>
<td>0.887264</td>
<td>0.741535</td>
<td>0.741600</td>
<td>100.009%</td>
</tr>
<tr>
<td>3.0</td>
<td>0.892979</td>
<td>0.712073</td>
<td>0.712667</td>
<td>100.083%</td>
</tr>
<tr>
<td>3.5</td>
<td>0.899747</td>
<td>0.690910</td>
<td>0.692000</td>
<td>100.158%</td>
</tr>
<tr>
<td>4.0</td>
<td>0.906402</td>
<td>0.674970</td>
<td>0.6765</td>
<td>100.227%</td>
</tr>
</tbody>
</table>

Fig. 3.8 Values for the gamma function $\Gamma^k (1 + \frac{1}{k})$ as used in the velocity distribution function.
Also shown in fig. 3.8 is an approximation of the gamma function by means of the analytical expression:

\[ G = 0.568 + \frac{0.434}{k} \quad (3.15) \]

This formula can easily be handled by pocket calculators in energy output calculations. The accuracy of the approximation is within 0.2% for \( 1 < k < 3.5 \) as shown in the last column of fig. 3.8.

In the expressions for \( f(V) \) and \( F(V) \) the ratio \( \frac{V}{V} \) often appears. Calling this the reduced wind velocity \( x \):

\[ x = \frac{V}{V} \quad (3.16) \]

the expressions (3.11) and (3.12) reduce to:

\[ F(x) = 1 - \exp \left(-r^k \left(1 + \frac{1}{k}\right) x^k\right) \quad (3.17) \]

and

\[ f(x) = k r^k \left(1 + \frac{1}{k}\right) x^{k-1} \exp \left(-r^k \left(1 + \frac{1}{k}\right) x^k\right) \quad (3.18) \]

When deriving (3.18) directly from (3.12) it should be borne in mind that:

\[ \int f(V) \, dV = \int_0^\infty f \left(\frac{V}{V}\right) \, d \frac{V}{V} = 1 \]

or

\[ \overline{f}(V) = f \left(\frac{V}{V}\right) \]

The graphs of \( F(x) \) and \( f(x) \) are shown in fig. 3.9 and fig. 3.10.
Fig. 3.9 The Weibull cumulative distribution function $F(x)$ as a function of the dimensionless wind speed $x = \frac{V}{\bar{V}}$ for different values of the Weibull shape parameter $k$. 
In the $F(x)$ graph for example we can see that in a $k=2$ wind regime during more than 95% of the time the wind speed will be below twice the average wind speed. In the $f(x)$ graph we can see that the most frequent wind speed in a $k=1.5$ wind regime has a value of about half the average wind speed. The value $f(x) = 0.67$ in this example indicates that wind speeds in an interval with a width of, say, $\overline{v}/10$ around this most frequent wind speed occur for a time fraction of $0.67/10$, or 6.7% of the time.

3.4.3 Estimation of the Weibull parameters from given data

The Weibull distribution shows its usefulness when the wind data of one reference station are being used to predict the wind regime in the surroundings of that station. The idea is that only annual or monthly average wind speeds are sufficient to predict the complete frequency distribution of the year or the month. This section deals with methods to extract the Weibull parameters $k$ and $c$ from a given set of data. Three methods are described:

1. Weibull paper
2. Standard-deviation analysis
3. Energy pattern factor analysis

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In principle it would be possible to construct the dimensionless velocity frequency curve or the cumulative distribution, to draw these curves in fig. 3.10 or fig. 3.9 and to "guess" the Weibull shape factor from their position. Comparing curves is a rather awkward business, however, so it is preferred to transform them into straight lines for comparison purposes.

The so-called Weibull paper is constructed in such a way that the cumulative Weibull distribution becomes a straight line, with the shape factor $k$ as its slope, as shown in fig. 3.12. Expression (3.7) can be rewritten as:
Fig. 3.10 The dimensionless Weibull wind speed frequency curve as a function of the dimensionless wind speed $x = \frac{V}{\bar{V}}$ for different values of the Weibull shape factor $k$. 
\[ (1 - F(V))^{-1} = \exp \left[ \frac{V}{c}^k \right] \] (3.19)

Taking the natural logarithm twice at both sides gives:

\[ \ln \ln (1 - F(V))^{-1} = k \ln V - k \ln c \] (3.20)

The horizontal axis of the Weibull paper now becomes \( \ln V \), while at the vertical axis \( \ln \ln (1 - F(V))^{-1} \) is placed. The result is a straight line with slope \( k \).

For \( V = c \) one finds: \( F(c) = 1 - e^{-1} = 0.632 \) and this gives an estimate for the value of \( c \), by drawing a horizontal line at \( F(V) = 0.632 \). The intersection point with the Weibull line gives the value of \( c \).

The practical procedure to find the Weibull shape factor from a given set of data starts with establishing the cumulative distribution of the data, as shown in fig. 3.6. We shall use these data, taken in Praia, Cape Verdian Islands (June 1975), in our example below.

The cumulative distribution refers to the total number of hours during which the wind speed was below a given value. If the number of hours in a specific interval is included in the cumulative number of hours belonging to that interval (as we did in fig. 3.6), then it is clear that we refer to the upper value of the interval for our calculations. The procedure now consists of plotting the percentages of the cumulative distribution as a function of the upper boundaries of their respective intervals on the Weibull paper. The result will be a number of dots lying more or less on a straight line. In case the line is really straight, the distribution perfectly fits to the Weibull distribution. In many cases, however, the line will be slightly bent. Then the linearization should be focussed on the wind speed interval that is most interesting for our wind energy applications, i.e. between \( 0.7 \times V \) and \( 2 \times V \).
The value of \( k \) is found by measuring the slope of the line just drawn. In fig. 3.12 we find an angle of 74° with the horizontal, so 
\[ k = \tan 74° = 3.49. \]
To facilitate the angle measurement, a simple geometrical operation can be performed: draw a second line through the "+", marked "k-estimation point", and perpendicular to the Weibull line. The intersection of this second line with the linear \( k \)-axis on top of the paper gives the desired \( k \)-value. In fig. 3.12 we find \( k \approx 3.5 \). The \( c \)-value, if required, is simply the intersection of the Weibull line with the dotted line, marked "c-estimation". In our example \( c = 8.3 \) m/s.

Preferably this procedure should be applied, not to the data of one month only, but to those of a number of years. If monthly \( k \)-values are required then the use of wind data from a number of identical months of subsequent years will give more reliable results.

STANDARD-DEVIATION ANALYSIS

With the expression for the standard deviation:

\[
\sigma = \sqrt{\int_0^\infty (V - \overline{V})^2 f(V) \, dV}.
\]  
(3.6)

and the expression for \( f(V) \)

\[
f(V) = \frac{k}{c} \left( \frac{V}{c} \right)^{k-1} \exp \left[ - \left( \frac{V}{c} \right)^k \right]
\]
(3.8)

one can find the following expression for \( \sigma \):

\[
\sigma = c \sqrt{\Gamma \left( 1 + \frac{2}{k} \right) - \Gamma^2 \left( 1 + \frac{1}{k} \right)}
\]
(3.21)

or with \( \overline{V} = c \Gamma \left( 1 + \frac{1}{k} \right) \):

\[
\frac{\sigma}{\overline{V}} = \sqrt{\frac{\Gamma \left( 1 + \frac{2}{k} \right) - \Gamma^2 \left( 1 + \frac{1}{k} \right)}{\Gamma \left( 1 + \frac{1}{k} \right)}}
\]
(3.22)
Fig 3.12 The cumulative velocity distribution of the month June 1975, measured in Praia (Cape Verdian Islands), plotted versus the upper boundary of the respective wind speed intervals, to yield the value of the Weibull shape factor $k$. 

Weibull probability paper for wind energy studies.
Cumulative distribution function $F(v)$ versus wind speed $v$.

Wind Energy Group, Dept. of Physics, Univ. of Technology, Eindhoven, Netherlands.
This function $\frac{\sigma}{\bar{V}}(k)$ is shown in fig. 3.13 [13].

So if the standard deviation of the distribution is calculated with

$$
\sigma^2 = s^2 = \frac{\Sigma(V_n)^2 - (\Sigma V_n)^2}{N}
$$

then the corresponding $k$-value can be found from fig. 3.13.

The data for Praia in fig. 3.1 result in:

$$
\sigma = 2.511
\quad \quad \quad \quad \quad \frac{\sigma}{\bar{V}} = 0.342 + k = 3.2
\quad \quad \quad \quad \quad \bar{V} = 7.35
$$

Fig. 3.13 The relative standard deviation of a Weibull distribution as a function of the Weibull shape factor $k$. 
ENERGY PATTERN FACTOR

The energy pattern factor $k_E$ is defined by Golding [2] as:

$$k_E = \frac{\text{total amount of power available in the wind}}{\text{power calculated by cubing the mean wind speed}} \quad (3.23)$$

Realizing that the power density of the wind is given by

$$\frac{P(V)}{A} = \frac{1}{2} \rho v^3 \quad [W/m^2] \quad (3.24)$$

then the total amount of energy per m$^2$ available in the wind in a period of $T$ seconds is equal to:

$$T \int_{0}^{\infty} \frac{1}{2} \rho v^3 f(v) \, dv \quad [J/m^2] \quad (3.25)$$

whereas the energy calculated by cubing the mean wind speed is equal to

$$\frac{1}{2} \rho \bar{v}^3 T \quad [J/m^2] \quad (3.26)$$

Using the Weibull probability density function $f(V)$ in (3.24) results, after some calculations, in

$$k_E = \frac{\Gamma(1 + \frac{3}{k})}{\Gamma^3(1 + \frac{1}{k})} \quad (3.27)$$

This function is shown in fig. 3.14 [13].

The energy pattern factor of a given set of $N$ hourly data $V_n$ can be determined with:

$$k_E = \frac{\frac{1}{N} \sum_{n=1}^{N} V_n^3}{\left(\frac{1}{N} \sum_{n=1}^{N} V_n^3\right)} \quad (3.28)$$

When this value is determined the Weibull shape parameter is easily found in fig. 3.14.
Fig. 3.14 The energy pattern factor of a Weibull wind speed distribution as a function of the Weibull shape factor $k$.

When carrying out the computation of $k_E$ for the original data of Praia in fig. 3.1 with formula 3.28 (which is a lengthy procedure for which programmable calculators are very useful) then the result is:

$$k_E = 1.356$$

Plotting this value in fig. 3.14 gives us the corresponding Weibull shape factor:

$$k = 3.2$$
The deviation from the value $k = 3.5$, as derived from the Weibull paper, can probably be explained by the fact that the distribution is not perfectly matching the ideal Weibull shape.

The energy pattern factor $k_E$ itself is utilized to indicate the wind potential of a given site by means of its power density:

$$\frac{P}{A} = k_E \cdot \frac{1}{2} \rho \bar{v}^3 \text{ W/m}^2$$  \hspace{1cm} (3.29)

In some cases very detailed measurements are carried out which even give the power density and other parameters as a function of the wind direction. This is shown in fig. 3.15 below, taken from Cherry [11]. In this table the power density is indicated as the Wind Energy Flux (WEF) and denoted by $\mathcal{F}$.

---

**Fig. 3.15** The wind speed and direction frequency distribution for Wellington Airport, 1960–1972, at 14.3 m above ground level.
4. Rotor Design*

4.1 General

This chapter discusses the design of horizontal-axis wind rotors, in which lift forces on airfoils are the driving forces. The design of a wind rotor consists of two steps: (1) the choice of basic parameters, such as the number of blades, the radius of the rotor, the type of airfoil, the tip speed ratio and (2) the calculations of the blade setting angle $\beta$ and the chord $c$ at each position along the blade. In this chapter both steps will be discussed, with an emphasis on the calculation of $\beta$ and $c$.

Before going into the details of how to calculate forces on airfoils, a general description of the behaviour of horizontal axis wind rotors will be presented, in which power, torque and speed play a major role.

4.2 Power, torque and speed

A wind rotor can extract power from the wind because it slows down the wind - not too much, not too little. At standstill the rotor obviously produces no power and at very high rotational speeds the air is more or less blocked by the rotor, and again no power is produced. In between these extremes there is an optimal rotational speed where the power extraction is at a maximum. This is illustrated in fig. 4.1.

* The sections 4.1 - 4.4 are based upon the SWD publication "Rotor Design", by W.A.M. Jansen and P.T. Smulders [16]. Section 4.5 is based upon a similar calculation by Sørensen in his "Renewable Energy" [17]. Sections 4.6 - 4.12 are based on the (Dutch) thesis of K. Heil on the behaviour of horizontal axis wind rotors [18] and the aerodynamic theory as developed by Wilson, Lissaman and Walker [5].
It is often also interesting to know the torque-speed curve of a wind rotor, for example when coupling a rotor to a piston pump with a constant torque. The power $P$ (W), the torque $Q$ (Nm) and the rotational speed $\Omega$ (rad/s) are related by a simple law:

$$P = Q \times \Omega$$  \hspace{1cm} (4.1)

With this relation fig. 4.2 is found from fig. 4.1:
It may be concluded that, because $Q = \frac{P}{\Omega}$, the torque is equal to the tangent of a line through the origin and some point of the $P-\Omega$ curve. This is why the maximum of the torque curve is reached at lower speeds than the maximum of the power curve (points 2 and 3 in figs. 4.1 and 4.2).

If the wind speed increases, power and torque increase, so for each wind speed a separate curve has to be drawn, both for power and for torque (fig. 4.3).

Fig. 4.3 The power and torque of a wind rotor as a function of rotational speed for different wind speeds
These groups of curves are rather inconvenient to handle as they vary with the wind speed $V$, the radius $R$ of the rotor, and even the density $\rho$ of the air. Power, torque and speed are made dimensionless with the following expressions:

\[
\begin{align*}
\text{power coefficient} & \quad C_p = \frac{p}{\frac{1}{2} \rho A V^3} \quad (4.2) \\
\text{torque coefficient} & \quad C_Q = \frac{Q}{\frac{1}{2} \rho A V^2 R} \quad (4.3) \\
\text{tip speed ratio} & \quad \lambda = \frac{\Omega R}{V} \quad (4.4)
\end{align*}
\]

with: rotor area $A = \pi R^2$

Substitution of these expressions in (1) gives:

\[
C_p = C_Q \cdot \lambda \quad (4.5)
\]

The immediate advantage is that the behaviour of rotors with different dimensions and at different wind speeds can be reduced to two curves: $C_p-\lambda$ and $C_Q-\lambda$.

In fig. 4.4 the typical $C_p-\lambda$ and $C_Q-\lambda$ curves of a multibladed and a two-bladed wind rotor are shown.
Fig. 4.4 Dimensionless power and torque curves of two wind rotors as a function of tip speed ratio

One significant difference between the rotors shown in fig. 4.4 becomes clear: multibladed rotors operate at low tip speed ratios and two or three-bladed rotors operate at high tip speed ratios. Note that their maximum power coefficient (at the so-called design tip speed ratio $\lambda_d$) does not differ all that much, but that there is a considerable difference in torque, both in starting torque (at $\lambda = 0$) and in maximum torque.

An empirical formula to estimate the starting torque coefficient of a rotor as a function of its design tip speed ratio is:

$$c_{Q,\text{start}} = \frac{0.5}{\lambda^2} \quad (4.6)$$
4.3 Airfoils: lift and drag

After having discussed the rotor as a whole, we shall turn to the behaviour of the blades by describing the lift and drag forces on the airfoil-shaped blades.

In fact, not only on airfoils, but on all bodies placed in a uniform flow, a force is exerted, of which the direction is generally not parallel to the direction of the undisturbed flow. The latter is crucial, because it explains why a part of the force, called the lift, is perpendicular to the direction of the undisturbed flow. The other part of the force, in the direction of the undisturbed flow, is called the drag. For an irregular body and for a smooth airfoil these forces are shown in fig. 4.5.

Fig. 4.5 A body placed in a uniform flow experiences a force \( F \) in a direction that is generally not parallel to the undisturbed flow. The actual direction of \( F \), and therefore the sizes of its two components, lift and drag, strongly depend on the shape of the body.
In physical terms the force on a body (such as an airfoil) is caused by the changes in the flow velocities (and direction) around the airfoil. On the upper side of the airfoil (see fig. 4.5) the velocities are higher than on the bottom side. The result is that the pressure on the upper side is lower than the pressure on the bottom side, hence the creation of the force $F$.

In describing the lift and drag properties of different airfoils, reference is usually being made to the dimensionless lift and drag coefficients, which are defined as follows:

\[
\text{lift coefficient } C_l = \frac{L}{\frac{1}{2} \rho V^2 A} \tag{4.7}
\]

\[
\text{drag coefficient } C_d = \frac{D}{\frac{1}{2} \rho V^2 A} \tag{4.8}
\]

with:  
\( \rho \) : density of air (kg/m\(^3\))  
\( V \) : undisturbed wind speed (m/s)  
\( A \) : projected blade area (chord * length) (m\(^2\))

These dimensionless lift and drag coefficients are measured in wind tunnels for a range of angles of attack $\alpha$. This is the angle between the direction of the undisturbed wind speed and a reference line of the airfoil. For a curved plate the reference line is simply the line connecting leading and trailing edge, while for an airfoil it is the line connecting the trailing edge with the center of the smallest radius of curvature at the leading edge (fig. 4.11).

The values of $C_p$ and $C_d$ of a given airfoil vary with the wind speed or, better, with the Reynolds number $Re$. The Reynolds number is a vital dimensionless parameter in fluid dynamics and, in the case of an airfoil, is defined as $Re = Vc/\nu$, with $V$ the undisturbed wind speed, $c$ the characteristic length of the body (here the chord of the airfoil) and $\nu$ the kinematic viscosity of the fluid (for air at 20° C the value of $\nu$ is 15*10^{-6} m\(^2\)/s).
In the following we shall neglect the influence of the Reynolds number, realizing that it has a second-order effect, and we shall assume that we possess the $C_l-\alpha$ and $C_d-\alpha$ curves for the proper value of Re.

An example of a $C_l-\alpha$ and a $C_l-C_d$ curve (the latter with $\alpha$ as a parameter) is shown in fig. 4.6.

![Diagram](image)

Fig. 4.6 The lift and drag coefficient of a given airfoil for a given Reynolds number.

The tangent to the $C_l-C_d$ curve drawn from the origin indicates the angle of attack with the minimum $C_d/C_l$ ratio. This ratio strongly determines the maximum power coefficients that can be reached, particularly at high tip speed ratios, as explained in the next paragraph.
The values of $\alpha$ and $C_l$ at minimum $C_d/C_l$ ratio are important parameters in the design process. The values for some airfoils are given in the table below (fig. 4.7).

<table>
<thead>
<tr>
<th></th>
<th>$C_d/C_l$</th>
<th>$\alpha$</th>
<th>$C_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat plate</td>
<td>0.1</td>
<td>5°</td>
<td>0.8</td>
</tr>
<tr>
<td>Curved plate (10% curvature)</td>
<td>0.02</td>
<td>3°</td>
<td>1.25</td>
</tr>
<tr>
<td>Curved plate with tube on concave side</td>
<td>0.03</td>
<td>4°</td>
<td>1.1</td>
</tr>
<tr>
<td>Curved plate with tube on convex side</td>
<td>0.2</td>
<td>14°</td>
<td>1.25</td>
</tr>
<tr>
<td>Airfoil NACA 4412</td>
<td>0.01</td>
<td>4°</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Fig. 4.7 Typical values of the drag-lift ratio $C_d/C_l$ and of $\alpha$ and $C_l$ for a number of airfoils. The curvature of the curved plate profile is defined as the ratio of its projected thickness and its chord.

What is the role of the lift and drag forces in the behaviour of a blade of a horizontal-axis wind rotor? To answer that question we must examine the wind speeds on a cross section of the blade, looking from the tip to the root of the blade (fig. 4.8).
Fig. 4.8 The wind speed $W$ seen by a cross section of the blade at a distance $r$ from the shaft is the vectorial sum of a component in the direction of the wind speed and a component in the rotor plane.

We see can that the relative wind speed $W$ which is seen by the blade, is composed of two parts:

1. The original wind speed $V$, but slowed down to a value $(1-a) V$ as a result of the power extraction.

2. A wind speed due to the rotational movement of the blade in the rotor plane. The value of this wind speed is slightly larger than $\Omega r$, the rotational speed of the blade at the cross section. The slight increase with respect to $\Omega r$ is caused by the rotation of the wake behind the rotor (see section 4.7).
The angle between the relative wind speed \(W\) and the rotor plane is \(\phi\). The lift force \(L\), due to the action of the wind speed \(W\) on the blade cross section, is by definition perpendicular to \(W\). As a result the angle between \(L\) and the rotor plane is \(90° - \phi\) and the forward component of the lift in the rotor plane (the driving force of the wind rotor) is equal to \(L \sin \phi\). By the same token the drag force in the rotor plane measures \(D \cos \phi\). In a situation of constant rotational speed these two components are just equal in value, but of opposite sign.

4.4 The maximum power coefficient

It has been shown by Betz (1926), with a simple axial momentum analysis (see 4.6), that the maximum power coefficient for a horizontal axis type wind rotor is equal to \(\frac{16}{27}\) or 59.3\%. This, however, is the power coefficient of an ideal wind rotor with an infinite number of (zero-drag) blades. In practice there are three effects which cause a further reduction in the maximum attainable power coefficient, namely:

1. The rotation of the wake behind the rotor.
2. The finite number of blades.
3. \(C_d/C_l\) ratio is not zero.

The creation of a rotating wake behind the rotor can be understood by imagining oneself as moving with the wind towards a multibladed wind rotor at standstill (fig. 4.9). The passage of the air between the rotor blades causes the blades to start moving to the left (in this example), but the air flow itself is deflected to the right (in fact this deflection causes the lift). The result is a rotation of the wake, implying extra kinetic energy losses and a lower power coefficient.
Fig. 4.9 The creation of a rotating wake behind a wind rotor.

For wind rotors with higher tip speed ratios, i.e. with smaller blades and a smaller flow angle $\phi$ (as we will see later), the effect of wake rotation is much smaller. For infinite tip speed ratios the Betz coefficient could be reached, were it not for the other two effects which come into play.

A finite number of blades, instead of the ideal infinite blade number, causes an extra reduction in power, particularly at low tip of the blade: the higher pressure at the lower side of the airfoil and the lower pressure at the upper side are "short circuited" at the tip of the blade, causing a crossflow around the tip, hence a decrease in pressure difference over the airfoil, and a lift force approaching zero at the tip itself. The length-width ratio of the whole blade determines the influence of this tip loss, the higher this ratio the lower the tip losses. To design a rotor with a given tip speed ratio, one can choose between many blades with a small chord width or less blades with a larger chord. With
this in mind, it will be clear that for a given tip speed ratio, a rotor with less blades will have larger tip losses. Since the chords become smaller for high tip speed ratios, this effect is smaller for higher tip speed ratios (fig. 4.10).

The last effect is the drag of the profile, characterized by the $C_d/C_l$ ratio of the airfoil. This causes a reduction of the maximum power coefficient which is proportional to the tip speed ratio and to the $C_d/C_l$ ratio. The result is shown in fig. 4.10. It must be stressed that these curves are not $C_p-\lambda$ curves, but $C_{p_{max}}-\lambda$ curves. They show for each $\lambda$ the maximum attainable power coefficient, with the blade number and the $C_d/C_l$ ratio as a parameter.
Fig. 4.10 The influence of blade number $B$ and drag/lift ratio $C_d/C_l$ on the maximum attainable power coefficient for each tip speed ratio. Taken from Rotor Design [16] after combining four different graphs.
4.5 Design of the rotor

The design of the rotor consists in finding both values of the chord \( c \) and the setting angle \( \beta \) (fig. 4.11) of the blades, at a number of positions along the blade. The calculations that follow are valid for a rotor operating at its maximum power coefficient, also called the "design" situation of the rotor. The values of \( \lambda \), \( C_1 \) and \( \alpha \) in this situation are referred to as \( \lambda_d \), \( C_{1d} \) and \( \alpha_d \).

![Diagram of rotor design showing angles and forces](image)

**Fig. 4.11** The angle of attack \( \alpha \) and the setting angle \( \beta \) of the blade of a wind rotor

The values for the following parameters must be chosen beforehand:

- \( R \) : the radius
- Rotor \( \lambda_d \) : the design tip speed ratio
- \( B \) : number of blades
- \( C_{1d} \) : design lift coefficient
- Airfoil \( \alpha_d \) : corresponding angle of attack
The radius of the rotor must be calculated with the required energy output $E$ in a year (or in a critical month), given the average local wind speed $V$ and its distribution. A simple approximation for water pumping windmills is given by (see section 2.1):

$$E = 0.1 \times \pi R^2 \times \bar{V}^3 \times T \quad [\text{Wh}]$$

(4.9)

with $T$: period length in hours.

This approximation is reasonable in situations where a design wind speed has been chosen which is equal to the average wind speed: $V_d = V$. For electricity generating wind turbines the factor 0.1 can be increased to 0.15, or sometimes to 0.2 or higher for very efficient machines.

The choices of $\lambda_d$ and $B$ are more or less related, as the following guidelines suggest:

<table>
<thead>
<tr>
<th>$\lambda_d$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6 - 20</td>
</tr>
<tr>
<td>2</td>
<td>4 - 12</td>
</tr>
<tr>
<td>3</td>
<td>3 - 6</td>
</tr>
<tr>
<td>4</td>
<td>2 - 4</td>
</tr>
<tr>
<td>5 - 8</td>
<td>2 - 3</td>
</tr>
<tr>
<td>8 - 15</td>
<td>1 - 2</td>
</tr>
</tbody>
</table>

Fig. 4.12 Guidelines for the choice of the design tip speed ratio and the number of blades.
The type of load will determine $\lambda_d$: water pumping windmills driving piston pumps have $1 < \lambda_d < 2$ and electricity generating wind turbines usually have $4 < \lambda_d < 10$.

The airfoil data are selected from fig. 4.7. Four formulas now describe the required information about $\beta$ and $c$:

- **Chord**
  \[ c = \frac{8\pi r}{BC \lambda_d} (1 - \cos \phi) \quad (4.10) \]

- **Blade setting angle**
  \[ \beta = \phi - \alpha \quad (4.11) \]

- **Flow angle**
  \[ \phi = \frac{2}{3} \arctan \frac{1}{\lambda_r} \quad (4.12) \]

- **Local design speed**
  \[ \lambda_{rd} = \lambda_d \frac{r}{R} \quad (4.13) \]

The design procedure will be described with the help of an example, in this case of a rotor designed by A. Kragten at the Eindhoven University of Technology, the Netherlands, as a part of the SWD programme [19]. The rotor is designed to drive a reciprocating piston pump.

- $R = 1.37 \text{ m}$
- $B = 6$
- $\lambda_d = 2$
- $C_{l1} = 1.1$ Curved plate profile (10% curvature)
- $\alpha_d = 4^\circ$ with tube in concave side (fig. 4.7)
The procedure is straightforward if it is decided to keep the lift coefficient at a constant value of \( C_{ld} \). In that case, a varying chord \( c \) and varying setting angle \( \beta \) will result. If it is wished to design a blade with a constant chord (for ease of production for example) then the lift coefficient will vary along the blade. We shall discuss these two possibilities, keeping in mind that many other alternatives exist.

4.5.1 Constant lift coefficient

The procedure consists of calculating the chord \( c \) and setting angle \( \beta \) at a number of positions along the blade, each with a distance \( r \) to the rotor axis and a local design speed ratio \( \lambda_{rd} \). In our case, four positions are chosen and for each position the relevant parameters are calculated with the formulas (4.10) to (4.13) and presented in fig. 4.13 and fig. 4.14.

<table>
<thead>
<tr>
<th>position</th>
<th>( r(m) )</th>
<th>( \lambda_{rd} )</th>
<th>( \phi^\circ )</th>
<th>( \alpha^\circ )</th>
<th>( \beta^\circ )</th>
<th>( c(m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.34</td>
<td>0.5</td>
<td>42.3</td>
<td>4</td>
<td>38.3</td>
<td>0.337</td>
</tr>
<tr>
<td>2</td>
<td>0.68</td>
<td>1.0</td>
<td>30.0</td>
<td>4</td>
<td>26.0</td>
<td>0.347</td>
</tr>
<tr>
<td>3</td>
<td>1.03</td>
<td>1.5</td>
<td>22.5</td>
<td>4</td>
<td>18.5</td>
<td>0.298</td>
</tr>
<tr>
<td>4</td>
<td>1.37</td>
<td>2</td>
<td>17.7</td>
<td>4</td>
<td>13.7</td>
<td>0.247</td>
</tr>
</tbody>
</table>

Fig. 4.13 Calculation of chord and setting angle for a six-bladed rotor \( \phi \ 2.74 \text{ m} \) with a constant lift coefficient.

In fig. 4.14 it can be seen that the chord of the blade is continuously varying along the blade. Also the setting angle varies, and the "twist" does not vary linearly along the blade. Both factors prevent an easy manufacture of this blade, and it is natural to look for ways to deviate from this shape without sacrificing too much of its performance. One approach is the constant chord blade.
Fig. 4.14 Blade shape and setting angles at four positions along the blade
4.5.2 Constant chord

Looking at formula (4.10) it can be seen that, with a constant chord \( c \), the lift coefficient at the different positions along the blade will vary:

\[
C_L = \frac{8 \pi r}{B \cdot c} (1 - \cos \phi)
\]  

(4.14)

Because variations in lift coefficient can only be accomplished via variations in the angle of attack, a fifth relation is needed in addition to our set of four equations, (4.10) to (4.13). The relation is:

\[
C_L = C_L(\alpha)
\]  

(4.15)

The \( C_L(\alpha) \) graph for the profile in our example is given in fig. 4.15.

Fig. 4.15 The lift coefficient of a curved plate (10% curvature) with a tube on the concave side
In the case of the rotor in our example, the dimensions of the blade were dictated by the standard sheet sizes: six blades had to be cut from a sheet of 1 x 2 m with a minimum loss. As a result, the uncurved blade measures 0.333 x 1 m, and the curved blade (10% curvature) has a chord of 0.324 m.

The positions chosen for the calculation are the positions of the 3 struts to fix the blade to the tube and are presented in fig. 4.16 below. The final shape of the blade is given in fig. 4.17.

<table>
<thead>
<tr>
<th>position</th>
<th>r(m)</th>
<th>( \lambda_{rd} )</th>
<th>( \phi )</th>
<th>c(m)</th>
<th>( C_L )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>chosen ( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.50</td>
<td>0.73</td>
<td>35.9°</td>
<td>0.324</td>
<td>1.23</td>
<td>6.4°</td>
<td>29.5°</td>
<td>27°</td>
</tr>
<tr>
<td>2</td>
<td>0.86</td>
<td>1.26</td>
<td>25.7°</td>
<td>0.324</td>
<td>1.10</td>
<td>3.6°</td>
<td>22.1°</td>
<td>23°</td>
</tr>
<tr>
<td>3</td>
<td>1.22</td>
<td>1.78</td>
<td>19.6°</td>
<td>0.324</td>
<td>0.91</td>
<td>0.2°</td>
<td>19.3°</td>
<td>19°</td>
</tr>
</tbody>
</table>

Fig. 4.16 Calculation of lift coefficient, \( \alpha \) and \( \beta \) for the constant chord blade of the SWD 2740 six-bladed rotor.

Fig. 4.17 Blade shape and setting angles of a blade of the six-bladed wind rotor.
In fig. 4.16 it can be noted that the setting angle finally chosen differs from the theoretical angle. This is because it is very difficult to manufacture a curved plate blade with a non-linear twist.

So starting from the correct angle nearby the tip, a good compromise is sought so as to keep the same change in angle between positions 1 and 2 and between positions 2 and 3. Also integer values for the angles are chosen.

The performance of the constant chord rotor has been measured with a rotor model, scaled down to Ø 1.5 m, and tested in the outlet of an open wind tunnel, Ø 2.2 m. The resulting $C_p-\lambda$ curve is shown in fig. 4.18.
Fig. 4.18 $C_p$-$\lambda$ curve of the six-bladed SWD 2740 rotor (Ø 2.74 m) with curved plate profiles
4.6 Power from airfoils: the sail boat analogy

A good introduction to understanding the power extraction by means of airfoils is to analyze the behavior of a sailboat or sailing car. It is particularly useful to demonstrate the difference between drag-driven and lift-driven devices. The example is taken from Sørensen's opus "Renewable Energy" [17].

For our analysis we use the speeds and angles as illustrated in fig. 4.19. This figure is nearly identical to fig. 4.8, except that the wind direction now enters at an angle $\phi$ with respect to the normal of the plane in which the airfoil moves. Later on this angle will become the angle of yaw of a horizontal-axis wind rotor.

Fig. 4.19 Velocity components for an airfoil moving with a speed $U$ in a windspeed $V$. 
The power extracted by the airfoil from the wind is given by:

\[ P = F_u \ U \]  
(4.16)

with \( F_u \) being the force in the \( U \)-direction. This force is composed of contributions by the lift and the drag:

\[ F_u = L \ \sin \phi - D \ \cos \phi \]  
(4.17)

where lift and drag are given by (see section 4.3)

\[ L = C_L \ c \ b \ \frac{1}{2} \rho \ W^2 \]  
(4.18)

\[ D = C_D \ c \ b \ \frac{1}{2} \rho \ W^2 \]  
(4.19)

with \( c \): chord of sail (blade)

\( b \): span of sail (blade)

The relative velocity \( W \) can be expressed in \( U \) and \( V \):

\[ W^2 = V^2 + U^2 - 2UV \ \sin \delta \]  
(4.20)

whereas \( \phi \) and \( \delta \) are related as follows:

\[ \sin \phi = \frac{V \ \cos \delta}{W} \ \text{and} \ \cos \phi = \frac{U - V \ \sin \delta}{W} \]

When introducing the speed ratio \( \lambda = \frac{U}{V} \) analogous to the tip speed ratio \( \lambda \) the expression for the power becomes:

\[ P = c \ b \ \frac{1}{2} \rho \ V^3 \ \lambda \ \left\{ \sqrt{(1 + \lambda^2 - 2 \ \lambda \ \sin \delta)} \ast (C_L \ \cos \delta - C_D (\lambda - \sin \delta)) \right\} \]  
(4.21)

Note that the power in the air for the area covered by the airfoil itself is given by \( c \ b \ \frac{1}{2} \rho \ V^3 \).
Now we can distinguish two cases: drag propulsion and lift propulsion. Drag propulsion occurs when the lift coefficient of the airfoil is assumed to be zero. We can see in expression (4.20) that in this case the highest power is reached when \( \sin \delta = 1 \) or \( \delta = 90^\circ \), i.e. when the ship is simply being pushed by the wind. The power is still a function of \( \lambda \):

\[
P = c \, b \, \frac{1}{2} \rho \, V^3 \, \lambda \, (1 - \lambda) \, C_D \, (1 - \lambda)
\]  

(4.22)

of which a maximum is reached for \( \lambda = \frac{1}{3} \) (by taking \( dP/d\lambda = 0 \)). The maximum power found is equal to:

\[
P_{\text{max}} = \frac{4}{27} \, C_D \, c \, b \, \frac{1}{2} \rho \, V^3
\]  

(4.23)

In other words, with the highest value of \( C_D = 2 \) for a half cylinder, the maximum power is only about \( \frac{8}{27} \approx 30\% \) of the power in the wind reaching the area of the sail.

In the case of lift propulsion the situation is quite different. Now the highest value of the last term in (4.20) is attained for \( \delta = 0 \), i.e. when the wind direction is perpendicular to the direction of movement of the sailboat. In this situation the power becomes:

\[
P = c \, b \, \frac{1}{2} \rho \, V^3 \, \lambda \, \sqrt{1 + \lambda^2} \, (C_L - C_D \, \lambda)
\]  

(4.24)

We can reasonably approach this, via

\[
\sqrt{1 + \lambda^2} \approx \lambda \, \text{(within 2\% for } \lambda > 5)\]

\[
P = c \, b \, \frac{1}{2} \rho \, V^3 \, \lambda^2 \, (C_L - C_D \, \lambda)
\]  

(4.25)

A maximum is reached for:

\[
\lambda = \frac{2}{3} \frac{C_L}{C_D}
\]  

(4.26)
The resulting maximum power becomes:

\[ P_{\text{max}} = \frac{4}{27} \left( \frac{C_L}{C_D} \right)^2 C_L c b \frac{1}{2} \rho V^3 \]  \hspace{1cm} (4.27)

We may conclude that even with simple airfoils, having \( C_L/C_D = 10 \) and \( C_L = 1 \), outputs can be reached that are about fifty times higher compared to the powers with drag propulsion. It is important to note that in this case the power is extracted from an area equal to about \( 400/27 = 15 \) times the actual blade area.

This is exactly why two or three bladed wind rotors, with a relatively small blade area compared to their swept area (the so-called solidity ratio), still can extract power from the whole swept area.

4.7. Axial momentum theory

The first description of the axial momentum theory was given by Rankine in 1865 and was improved later by Froude. The theory provides a relation between the forces acting on a rotor and the resulting fluid velocities and predicts the ideal efficiency of the rotor. Later on Betz included rotational wake effects in the theory. Recently, Wilson, Lissaman and Walker have further analyzed the aerodynamic performance of wind turbines [5].

For our analysis we shall use the symbols as indicated in fig. 4.20 below. Note that for the undisturbed wind speed \( V_1 \) (= \( V_\beta \) of chapter 2) we shall later on use the notation \( V \).
Fig. 4.20 Schematic illustration of the parameters involved in the description of the axial momentum theory for a wind rotor.

The assumptions underlying the axial momentum theory are:

- incompressible medium
- no frictional drag
- infinite number of blades
- homogeneous flow
- uniform thrust over the rotor area
- non-rotating wake
- static pressure far before and far behind the rotor is equal to the undisturbed ambient static pressure.
Considering the stream tube indicated in fig. 4.18 the conservation of mass dictates:

\[ \rho A_1 V_1 = \rho A_2 V_2 \]  \hspace{1cm} (4.28)

The thrust force \( T \) on the rotor is given by the change in momentum of the incoming flow compared to the outgoing flow:

\[ T = \rho A_1 V_1^2 - \rho A_2 V_2^2 \]  \hspace{1cm} (4.29)

With (4.28) this becomes:

\[ T = \rho A \, \Delta V \, (V_1 - V_2) \]  \hspace{1cm} (4.30)

Also the thrust on the rotor can be expressed as a result of the pressure difference over the rotor area:

\[ T = (p^+ - p^-) A \]  \hspace{1cm} (4.31)

The pressures can be found with Bernoulli's equation:

before the rotor: \( p + \frac{1}{2} \rho V_1^2 = p^+ + \frac{1}{2} \rho V_{ax}^2 \) \hspace{1cm} (4.32)

behind the rotor: \( p^- + \frac{1}{2} \rho V_{ax}^2 = p + \frac{1}{2} \rho V_2^2 \)

This yields:

\[ p^+ - p^- = \frac{1}{2} \rho (V_1^2 - V_2^2) \]  \hspace{1cm} (4.33)

and the thrust becomes:

\[ T = \frac{1}{2} \rho A(V_1^2 - V_2^2) \]  \hspace{1cm} (4.34)

Equating (4.34) with (4.30) provides the important relation:

\[ V_{ax} = \frac{1}{2} (V_1 + V_2) \]  \hspace{1cm} (4.35)
We shall now introduce the **axial induction factor** \( a \) with:

\[
V_{ax} = V_1 (1 - a) \tag{4.36}
\]

Substitution of (4.35) gives us the down stream velocity:

\[
V_2 = V_1 (1 - 2a) \tag{4.37}
\]

The power absorbed by the rotor is equal to the change in kinetic power of the mass flowing through the rotor area:

\[
P = \frac{1}{2} \rho \, A \, V_{ax} \, \left( V_1^2 - V_2^2 \right) \tag{4.38}
\]

With (4.36) and (4.37) the expression for the power becomes:

\[
P = 4a (1 - a)^2 \frac{1}{2} \rho \, A \, V_1^3 \tag{4.39}
\]

The maximum value of \( P \) is reached for \( \frac{dP}{da} = 0 \) and this results in:

\[
a = \frac{1}{3} \tag{4.40}
\]

Substitution of this value in (4.39) gives us:

\[
P = \frac{16}{27} \times \frac{1}{2} \rho \, A \, V_1^3 \tag{4.41}
\]

The factor \( \frac{16}{27} \) is called the Betz-coefficient (1926), as we have mentioned in section 2.1, and represents the maximum fraction which an ideal wind rotor under the given conditions can extract from the flow. Note that this fraction is related to the power of an undisturbed flow arriving at area \( A \), whereas in reality only the undisturbed flow through area \( A_1 = A(1 - a) \) reaches the rotor. This means that the maximum efficiency related to the real mass flow through the rotor is equal to \( \frac{16}{27} \times \frac{3}{2} = \frac{8}{9} \).
The ideal model of a completely axial flow before and behind the rotor has to be modified when realising that a rotating rotor implies the generation of angular momentum (torque). This means that in reaction to the torque exerted by the flow on the rotor, the flow behind the rotor rotates in the opposite direction.

This rotation represents an extra loss of kinetic energy for the wind rotor, a loss that will be higher if the torque to be generated is higher. The conclusion is that slow-running wind rotors (with a low tip speed ratio and a high torque) experience more wake rotation losses than the high tip speed machines with low torque. This is the reason for the shape of the upper boundary of the curves in fig. 4.10, approaching the Betz-limit only for high \( \lambda \)-values.

For our analysis we shall use the annular stream tube model, in order to be able to incorporate variations of the parameters involved along the blade (fig. 4.21).

Fig. 4.21 The stream tube model, illustrating the rotation of the wake.
With a ring radius \( r \) and a thickness \( dr \), the cross sectional area of the annular tube becomes \( 2\pi r \, dr \). If we imagine ourselves moving along with the blades it can be shown (the proof will not be given here) that we may apply Bernoulli's equation to derive an expression for the pressure difference over the blades. Now the relative angular velocity increases from \( \Omega \) to \( \Omega + \omega \), while the axial components of the velocity remain unchanged. We find:

\[
\frac{p^+ - p^-}{\rho} = \frac{1}{2} \rho (\Omega + \omega)^2 \frac{r^2}{\rho^2} - \frac{1}{2} \rho \Omega^2 \frac{r^2}{\rho^2}
\]

or

\[
\frac{p^+ - p^-}{\rho} = \rho (\Omega + \frac{1}{2} \omega) \frac{\omega r^2}{\rho}
\]

(4.42)

The resulting thrust on the annular element of the rotor is:

\[
dT = \rho (\Omega + \frac{1}{2} \omega) \frac{\omega r^2}{\rho} 2\pi r \, dr
\]

(4.43)

or, by introducing the tangential induction factor \( a' \):

\[
a' = \frac{\frac{1}{2} \omega}{\Omega}
\]

(4.44)

the expression for the thrust becomes:

\[
dT = 4a' (1 + a') \frac{1}{2} \rho \Omega^2 \frac{r^2}{\rho^2} 2\pi r \, dr
\]

(4.45)

We may equate this expression to the similar expression (4.34) derived by the axial momentum theory, by introducing the axial interference factor \( a \) in (4.34) and looking at an annular cross section only \((A + 2\pi r \, dr \text{ and } v_1 + v)\):

\[
dT = 4a (1 - a) \frac{1}{2} \rho \frac{v^2}{2} 2\pi r \, dr
\]

(4.46)

the result is:

\[
\frac{a (1 - a)}{a'(1 + a')} = \frac{\Omega^2 r^2}{v^2} = \lambda_r^2
\]

(4.47)

We shall use this relation later.
Apart from an expression for the thrust on the rotor, it is possible to derive a relation for the torque exerted on the rotor. This is achieved by realizing that the conservation of angular momentum implies that the torque exerted must be equal to the angular momentum of the wake:

\[ dQ = \rho V_{ax} \left( 2\pi r dr \right) \omega r r \]

Introducing the axial and tangential induction factors with (4.36) (noting that \( V_1 + V \)) and (4.44), the expression for the torque on an annular element of the rotor becomes:

\[ dQ = 4a' (1 - a) \frac{\rho V}{\Omega} r r 2\pi r dr \]

The power generated is equal to \( dP = \Omega dQ \), so the total power is equal to:

\[ P = \int^{R}_{0} \Omega dQ \]

Introducing the local speed ratio \( \lambda_r \) with:

\[ \lambda_r = \frac{\Omega r}{V} \]

the power becomes:

\[ P = \frac{\rho A V^3}{\lambda^2} \int^{\lambda}_{0} a'(1 - a) \lambda^3 \lambda_r d \lambda_r \]

or the power coefficient \( C_p \) is equal to:

\[ C_p = \frac{8}{\lambda^2} \int^{\lambda}_{0} a'(1 - a) \lambda^3 \lambda_r d \lambda_r \]
The maximum value of \( a' (1 - a) \) can be found by using relation (4.47) to express \( a' \) in \( a \):

\[
a' = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4}{\lambda^2 \tau} a (1 - a)}
\]  

Substituting this expression in \( a' (1 - a) \) and taking the derivative equal to zero yields, after some manipulations:

\[
\lambda^2 \tau = \frac{(1 - a)(4a - 1)^2}{1 - 3a}
\]  

and this expression implies the following relation between \( a' \) and \( a \) (with 4.47):

\[
a' = \frac{1 - 3a}{4a - 1}
\]  

The last two expressions are tabulated below:

<table>
<thead>
<tr>
<th>( a )</th>
<th>( a' )</th>
<th>( \lambda^2 \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>0.26</td>
<td>5.5</td>
<td>0.073</td>
</tr>
<tr>
<td>0.27</td>
<td>2.375</td>
<td>0.157</td>
</tr>
<tr>
<td>0.28</td>
<td>1.333</td>
<td>0.255</td>
</tr>
<tr>
<td>0.29</td>
<td>0.812</td>
<td>0.374</td>
</tr>
<tr>
<td>0.30</td>
<td>0.500</td>
<td>0.529</td>
</tr>
<tr>
<td>0.31</td>
<td>0.292</td>
<td>0.754</td>
</tr>
<tr>
<td>0.32</td>
<td>0.143</td>
<td>1.154</td>
</tr>
<tr>
<td>0.33</td>
<td>0.031</td>
<td>2.619</td>
</tr>
<tr>
<td>0.333</td>
<td>0.003</td>
<td>8.574</td>
</tr>
<tr>
<td>0.3333</td>
<td>0.0003</td>
<td>27.206</td>
</tr>
<tr>
<td>( \frac{1}{3} )</td>
<td>0</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

Fig. 4.22 The values of \( a \) and \( a' \) for different local speed ratios \( \lambda^2 \tau \) of an optimized ideal wind rotor.
Via numerical integration we now can find values for the maximum power coefficient $C_p$ as a function of $\lambda$ via (4.53). The result is tabulated below and represents the upper boundary of the curves shown in fig. 4.9.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$C_{p_{\text{max}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.288</td>
</tr>
<tr>
<td>1.0</td>
<td>0.416</td>
</tr>
<tr>
<td>1.5</td>
<td>0.481</td>
</tr>
<tr>
<td>2.0</td>
<td>0.513</td>
</tr>
<tr>
<td>2.5</td>
<td>0.533</td>
</tr>
<tr>
<td>5.0</td>
<td>0.570</td>
</tr>
<tr>
<td>7.5</td>
<td>0.582</td>
</tr>
<tr>
<td>10.0</td>
<td>0.585</td>
</tr>
<tr>
<td>$\infty$</td>
<td>16/27</td>
</tr>
</tbody>
</table>

Fig. 4.23 The maximum attainable power coefficient to be extracted by an ideal wind rotor at a given tip speed ratio.

4.8 Blade element theory

The momentum theory as derived in section 4.7 cannot provide us with the necessary information on how to design the blades of a wind rotor. The blade element theory, combined with the momentum theory, will provide us with this information. Its approach is opposed to that of the momentum theory: the forces on each blade element are calculated with given fluid velocities.

The assumptions underlying the blade element theory are:
- there is no interference between adjacent blade elements along each blade
- the forces acting on a blade element are solely due to the lift and drag characteristics of the sectional profile of a blade element.

The procedure is to calculate the forces on each differential blade element and subsequently integrate them over the length of the blade, (and multiplying by the number of blades) to find expressions for the torque and thrust. We shall use the notations shown in fig. 4.22 (similar to fig. 4.10). We shall assume that each blade element moves in the same plane, i.e. the coning angle is zero.

Fig. 4.22 Wind speeds and forces acting on a blade element of a horizontal axis wind rotor.

The following expressions are used for the sectional lift and drag:

\[
\begin{align*}
    dL &= C_L \frac{1}{2} \rho W^2 c \, dr \\
    dD &= C_D \frac{1}{2} \rho W^2 c \, dr
\end{align*}
\]

(4.57)

The thrust and torque experienced by the blade element are:

\[
\begin{align*}
    dT &= dL \cos \phi + dD \sin \phi \\
    dQ &= (dL \sin \phi - dD \cos \phi) \times r
\end{align*}
\]

(4.58) (4.59)
with (4.57) and assuming that the rotor has $B$ blades the expressions for thrust and torque become:

$$
\begin{align*}
\frac{dT}{dr} &= B \frac{1}{2} \rho \omega^2 (C_l \cos \phi + C_d \sin \phi) c r dr \quad (4.60) \\
\frac{dQ}{dr} &= B \frac{1}{2} \rho \omega^2 (C_l \sin \phi - C_d \cos \phi) c r dr \quad (4.61)
\end{align*}
$$

### 4.9 Combination of momentum theory and blade element theory

For convenience of the reader we shall repeat the formulas derived by both theories:

**momentum theory**

$$
\begin{align*}
\frac{dT}{dr} &= 4a (1 - a) \frac{1}{2} \rho V^2 2\pi r dr \quad (4.46) \\
\frac{dQ}{dr} &= 4a (1 - a) \frac{1}{2} \rho V \Omega r 2\pi r dr \quad (4.49)
\end{align*}
$$

**blade element theory**

$$
\begin{align*}
\frac{dT}{dr} &= (C_l \sin \phi - C_d \cos \phi) \frac{1}{2} \rho \omega^2 B c r dr \quad (4.60) \\
\frac{dQ}{dr} &= (C_l \sin \phi - C_d \cos \phi) \frac{1}{2} \rho \omega^2 B c r dr \quad (4.61)
\end{align*}
$$

In order to combine the results from the blade element theory with those of the momentum theory we need an expression for the relative wind velocity $W$. This is found with fig. 4.22 or with fig. 4.23 below.

![Velocity diagram for a blade element of a horizontal axis wind rotor.](image)

Fig. 4.23 Velocity diagram for a blade element of a horizontal axis wind rotor.
From fig. 4.23 we may conclude that:

$$W = \frac{(1 - a) \cdot V}{\sin \phi} \quad (l + a') \quad \frac{\Omega \cdot r}{\cos \phi}$$  \hspace{1cm} (4.62)

and also:  $$\tan \phi = \frac{(1 - a) \cdot V}{(1 + a') \cdot \Omega \cdot r} \quad \frac{1 - a \cdot \frac{1}{\lambda r}}{1 + a'}$$  \hspace{1cm} (4.63)

If we introduce the local solidity ratio \(\sigma\) via

$$\sigma = \frac{B \cdot c}{2 \pi r}$$  \hspace{1cm} (4.64)

the results of the blade element theory transform into:

$$dT = (1 - a)^2 \quad \frac{\sigma \cdot C_1 \cdot \cos \phi}{\sin^2 \phi} \quad (1 + \frac{C_d}{C_1} \cdot \tan \phi) \quad \frac{\frac{1}{2} \rho \cdot V^2 \cdot 2 \pi r \cdot dr}{\Omega^2 \cdot r^2 \cdot r}$$  \hspace{1cm} (4.65)

$$dQ = (1 + a')^2 \quad \frac{\sigma \cdot C_1 \cdot \sin \phi}{\cos^2 \phi} \quad (1 - \frac{C_d}{C_1} \cdot \frac{1}{\tan \phi}) \quad \frac{\frac{1}{2} \rho \cdot \Omega^2 \cdot r^2 \cdot 2 \pi r \cdot dr}{\Omega^2 \cdot r^2 \cdot r}$$  \hspace{1cm} (4.66)

Combining (4.65) and (4.66) yields:

$$\frac{4a}{1 - a} = \sigma \cdot C_1 \quad \frac{\cos \phi}{\sin^2 \phi} \quad (1 + \frac{C_d}{C_1} \cdot \tan \phi)$$  \hspace{1cm} (4.67)

whereas (4.66), (4.49) and (4.63) give:

$$\frac{4a'}{1 + a'} = \sigma \cdot C_1 \quad \frac{\cos \phi}{\cos \phi} \quad (1 - \frac{C_d}{C_1} \cdot \frac{1}{\tan \phi})$$  \hspace{1cm} (4.68)

It is argued by some authors (Jansen [35], Wilson, Lissaman and Walker [5] and de Vries [20]) that the drag terms should be omitted from (4.67) and (4.68) because the profile drag does not induce velocities at the blade itself within the approximation of small blade chords. Other authors (Heil [18] and Griffith [21]), however, continue with the drag terms included. Here we shall omit the drag terms and then calculate the induction factors \(a\) and \(a'\) with:
These two relations (4.69 and 4.70 without drag or 4.67 and 4.68 with drag) together with (4.63) and the two expression for dT and dQ (4.65) and (4.66) determine the behaviour of the wind rotor. For actual calculations the \((C_l - \alpha)\) and \((C_d - \alpha)\) characteristics are needed and the fact that \(\phi = \alpha + \beta\) (4.11). Later on we shall see that corrections have to be made for tip losses and the fact that the blade number is finite.

Here we shall derive expressions for \(C_p\) and shall estimate the effect of profile drag on the \(C_p\). The expression for \(C_p\) is:

\[
C_p = \frac{1}{\frac{1}{2} \rho V^3 \pi R^2} \int_0^R \frac{\omega}{\varphi} dQ \tag{4.71}
\]

The expression (4.66) for the torque \(dQ\) can be transformed with (4.70) and (4.63) into:

\[
dQ = 4a' (1 - \alpha) \left(1 - \frac{C_d}{C_l} \frac{1}{\tan \phi}\right) \frac{1}{\lambda} \frac{1}{2} \rho \Omega^2 r^2 \pi r \, d\lambda \tag{4.72}
\]

Substituting (4.72) in (4.71) and realizing that

\[
r = \lambda \frac{R}{\lambda} \text{ and } dr = \frac{R}{\lambda} d\lambda
\]

we find the expression for \(C_p\):

\[
C_p = \frac{8}{\lambda^2} \int_0^\lambda a' (1 - \alpha) \lambda^3 \left(1 - \frac{C_d}{C_l} \frac{1}{\tan \phi}\right) d\lambda \tag{4.73}
\]
Remembering formula (4.53) for the wind rotor with \( C_D = 0 \) we see:

\[
C_p = C_p(C_D = 0) - \frac{8}{\lambda^2} \int_0^\lambda a' (1-a) \lambda^2 \frac{C_d}{C_1} \frac{1}{r} \frac{1}{\tan \phi} \, d\lambda_r \quad (4.74)
\]

With expressions (4.63) and (4.47) this becomes:

\[
C_p = C_p(C_D = 0) - \frac{8}{\lambda^2} \int_0^\lambda a (1-a) \lambda^2 \frac{C_d}{C_1} \frac{1}{r} \frac{1}{\tan \phi} \, d\lambda_r \quad (4.75)
\]

From table 4.20 we see that \( a = \frac{1}{3} \) for \( \lambda > 1 \) and if we assume that the \( C_d/C_1 \) ratio remains constant along the blade (which is true under optimum conditions only) we may rewrite (4.75) into:

\[
C_p = C_p(C_D = 0) - \frac{8}{\lambda^2} \int_0^\lambda \frac{C_d}{C_1} \frac{2}{9} \lambda^2 \frac{1}{r} \, d\lambda_r
\]

or

\[
C_p = C_p(C_D = 0) - \frac{16}{27} \frac{C_d}{C_1} \lambda \quad (4.76)
\]

This formula explains the shape of the curves in fig. 4.10 although it is only valid for an infinite number of blades. The effect of a finite blade number will be discussed in the next section.

4.10 Tip losses

In the preceding sections the rotor was assumed to be a so-called "actuator disk" possessing an infinite number of blades, with an infinitely small chord. Let us consider a real situation in which the number of blades is finite. The lift, as introduced in section 4.3, is generated by the pressure distribution around the blade due to the two-dimensional flow. On the upper side (see fig. 4.6) the pressure is below ambient, on the lower it is above ambient pressure. At the tip however this pressure difference leads to
secondary flow around the tip; the flow becomes three-dimensional and tries to equalize the pressure difference, thus reducing the lift. This effect is more pronounced as one approaches the tip. It results in a reduction of the torque on the rotor and so reduces the power output. The effect on $C_p$ is referred to as "tip losses".

There are a number of theories to calculate these tip losses, one more elaborate than the other. We shall only present here the results of the model developed by Prandtl. The essential idea of the Prandtl theory is that the velocities in the rotor plane, as seen by the blade and calculated with the aid of momentum theory, are altered by the disturbed flow near the tip. For this purpose Prandtl developed the following function $F$ (the derivation is outside the scope of this Introduction):

$$F = \frac{2}{\pi} \arccos \left\{ \exp \left( -\frac{1}{2} B \frac{R - r}{r \sin \phi} \right) \right\} \quad (4.77)$$

There are different opinions about how to introduce the tip loss factor $F$ in the rotor calculations. We shall utilize the method adopted by Wilson and Lissaman [5] who assume that the induction factors $a$ and $a'$ have to be multiplied with $F$, meaning that the axial velocity and tangential velocities in the rotor plane as seen by the blades are modified. It is assumed that these corrections only involve the momentum formulas.

The effect on the momentum formulas is as follows:

$$(4.46) + \frac{dT}{\rho} = 4aF (1 - aF) \frac{\rho}{2} V^2 2\pi r \, dr \quad (4.78)$$

$$(4.49) + \frac{dQ}{\rho} = 4a'F (1 - aF) \frac{\rho}{2} \Omega r^2 r \, 2\pi r \, dr \quad (4.79)$$

The results of the blade element theory remain unchanged:

$$dT = (1 - a)^2 \frac{\sigma C_1 \cos \phi}{\sin^2 \phi} (1 + \frac{C_d}{C_1} \tan \phi) \frac{\rho}{2} V^2 2\pi r \, dr \quad (4.65)$$

$$dQ = (1 - a)^2 \frac{\sigma C_1}{\sin \phi} (1 - \frac{C_d}{C_1} \frac{1}{\tan \phi}) \frac{\rho}{2} V^2 r \, 2\pi r \, dr \quad (4.66a)$$
The latter expression is found by substituting (4.63) in (4.66). The two expressions for the thrust result in:

\[ 4aF (1 - af) = (1 - a)^2 \frac{\sigma C_1 \cos \phi}{\sin^2 \phi} (1 + \frac{C_d}{C_1} \tan \phi) \tag{4.80} \]

and the torque expressions yield:

\[ 4a'F (1 - aF) = (1 - a)^2 \frac{\sigma C_1}{\sin \phi} (1 - \frac{C_d}{C_1} \tan \phi) \tag{4.81} \]

These two last relations, together with

\[ \tan \phi = \frac{1 - a}{1 + a'} \frac{1}{\lambda_r} \tag{4.63} \]
\[ \alpha = \phi - \beta \tag{4.10} \]

and the relations describing the forces on the profile, i.e. \( C_l(\alpha) \) and \( C_d(\alpha) \) together with the expression (4.77) for \( F \) describe the behaviour of the rotor.

### 4.11 Design for maximum power output

As we have seen in section 4.7 the design for maximum power output implies the following relation between \( a \) and \( a' \)

\[ a' = \frac{1 - 3a}{4a - 1} \tag{4.56} \]

Substituting this relation in (4.70) and eliminating the parameter "a" subsequently with (4.69) leads to the simple expression:

\[ \sigma C_1 = 4(1 - \cos \phi) \tag{4.82} \]

and with \( \sigma = \frac{B c}{2\pi r} \) this transforms into the formula we have used to design our blades in section 4.4:
\[ c = \frac{8\pi r}{B C_l} (1 - \cos \phi) \]  \hspace{1cm} (4.9)

in which case we used the optimum value \( C_{L_d} \) for \( C_l \).

We are still lacking the relation between \( \lambda_r \) and \( \phi \). This is found by substituting (4.82) into (4.69) and (4.70) and by applying:

\[ \lambda_r = \frac{1 - a}{1 + a'} \frac{1}{\tan \phi} \]  \hspace{1cm} (4.63)

The result is:

\[ \lambda_r = \frac{\sin \phi (2 \cos \phi - 1)}{(1 - \cos \phi)(2 \cos \phi + 1)} \]  \hspace{1cm} (4.83)

and this expression can be reduced to:

\[ \lambda_r = \frac{1}{\tan \frac{3}{2} \phi} \]  \hspace{1cm} (4.84)

This is similar to formula (4.11):

\[ \phi = \frac{2}{3} \arctan \frac{1}{\lambda_r} \]  \hspace{1cm} (4.11)

These results provided the basis for the design methods utilized earlier in section 4.4 (formulas 4.9 to 4.12).

It is interesting to calculate the effect of, for example, a constant chord blade on the other parameters, with the simple design formulas of this section.

For this purpose we assume the \((C_l - \alpha)\) graph to be linear:

\[ C_l = C_{l_0} + \frac{d C_l}{d \alpha} \alpha \]  \hspace{1cm} (4.85)

where \( C_{l_0} \) is the value of \( C_l \) at \( \alpha = 0 \).
For small angles of $\alpha$, i.e. below the stalling angle, this is a reasonable approximation for many profiles. For short we write:

$$C'_1 = \frac{d C_1}{d \alpha} \quad (4.86)$$

and the angle $\alpha$ becomes:

$$\alpha = \frac{C_1 - C_{10}}{C'_1} \quad (4.87)$$

The sectional lift coefficient is given by (4.9) so we can write (4.10) as:

$$\beta = \phi - \frac{8\pi r}{B c} \left(1 - \cos \phi\right) - C_{10} \quad (4.88)$$

with $r = \frac{R}{\lambda}$ and with (4.11) this becomes:

$$\beta = \phi - \frac{8\pi R}{B c \lambda} \frac{1 - \cos \phi}{C'_1} \frac{C_{10}}{\tan \frac{3}{2} \phi} \quad (4.89)$$

We can approximate the goniometrical part for small angles $\phi$ as follows ($\phi$ in radians):

$$\frac{1 - \cos \phi}{\tan \frac{3}{2} \phi} = \frac{1 - (1 - \frac{1}{2} \phi^2 + \frac{1}{24} \phi^4 - \ldots)}{\frac{3}{2} \phi + \frac{27}{24} \phi^3 + \ldots} = \frac{1}{3} \phi \quad (4.90)$$
We may conclude that the setting angle $\beta$ can be constant (easy to manufacture) if the following condition is met (for small flow angles $\phi$):

$$\frac{8\pi R}{B \cos \lambda C'_1} = 3$$  \hspace{1cm} (4.91)

and in that case the setting angle is equal to:

$$\beta = \frac{C_{10}}{C'_1} \text{ (radians)}$$  \hspace{1cm} (4.92)

Example:

NACA 4412 profile with: $C'_1 = \frac{1.0}{10^\circ} = \frac{1.0}{0.175} = 5.73$

and: $C_{10} = 0.4$

With (4.92) we find: $\beta = 4^\circ$

and: $\frac{R}{Bc\lambda} = 0.68$

When we want to design a three-bladed rotor with a design tip speed ratio of 6 and a diameter of 4 metres, this approximation leads to:

$$c = \frac{2}{3 \times 6} \times \frac{1}{0.68} = 0.16 \text{ m}$$

If we had designed the blade according to the procedure in section 4.4, with a varying chord and a varying setting angle, then the result would have been as follows (assuming $C_{1d} = 1$):
4.12 Calculating the rotor characteristics

Once we have designed a rotor according to the formulas in the previous sections (and probably changed the shape somewhat for easy manufacture) we would like to calculate the characteristics of the rotor. Here we shall not go into the details of how to write a computer program for this purpose, but just outline the method. It is by far not the only method available but it is a common one, introduced by Wilson and Lissaman [5].

We may assume that the following data of the rotor are available:

- radius \( R \)
- setting angles \( \beta(r) \)
- chords \( c(r) + \sigma(r) \)
- tip speed ratio \( \lambda \)
- number of blades \( B \)
- profile characteristics \( C_1(\alpha) \) and \( C_d(\alpha) \)

<table>
<thead>
<tr>
<th>position</th>
<th>( \beta )</th>
<th>( c )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2 ( R )</td>
<td>20.5(^\circ)</td>
<td>0.353 m</td>
<td>16.5(^\circ)</td>
</tr>
<tr>
<td>0.4 ( R )</td>
<td>9.1(^\circ)</td>
<td>0.231 m</td>
<td>5.1(^\circ)</td>
</tr>
<tr>
<td>0.6 ( R )</td>
<td>4.3(^\circ)</td>
<td>0.164 m</td>
<td>4.1(^\circ)</td>
</tr>
<tr>
<td>0.8 ( R )</td>
<td>1.8(^\circ)</td>
<td>0.125 m</td>
<td>4.0(^\circ)</td>
</tr>
<tr>
<td>( R )</td>
<td>0.3(^\circ)</td>
<td>0.101 m</td>
<td>4.0(^\circ)</td>
</tr>
</tbody>
</table>
The procedure now consists of finding values for the induction factors $a$ and $a'$ for a number of blade positions $r$. Because there is no analytical expression for the induction factors we have to use an iteration method with the following steps.

1. Choose a value for $r + \frac{\lambda}{R} = \frac{r}{R} \lambda$

2. Assume reasonable starting values for $a$ and $a'$ (e.g. $a = \frac{1}{3}$ and $a' = 0$).

3. Calculate $\phi$ with $\phi = \arctan \left( \frac{1 - a}{1 + a'} \right) \lambda$ \hspace{1cm} (4.63)

4. Calculate $\alpha$ with $\alpha = \phi - \beta$ \hspace{1cm} (4.11)

5. Calculate $C_1$ with the $C_1(\alpha)$ graph or table

6. Calculate $a$ with $\frac{4a}{1 - a} = \sigma C_1 \frac{\cos \phi}{\sin^2 \phi}$ \hspace{1cm} (4.69)

and $a'$ with $\frac{4a'}{1 + a'} = \sigma \frac{C_1}{\cos \phi}$ \hspace{1cm} (4.70)

7. Compare with the values of $a$ and $a'$ from 1. and iterate until the desired accuracy is attained.

8. Calculate the values of $C_d$, $dQ/dr$ and $dT/dr$, or directly the values of $dC_Q/dr$ and $dC_T/dr$.

When this procedure has been followed for a number of positions $r$ along the blade, then the total value of $C_T$, $C_Q$ and hence $C_p$ can be found with a numerical integration procedure. If tip losses are to be included then the formulas change accordingly, but the reader will discover that an extra loop has to be included (Heil [18]). Also, for values of $\alpha$ where stalling of the blade occurs, multiple solutions might occur (Wilson and Lissaman [5]).
5. **PUMPS**

5.1 **General**

The need to lift water is as old as mankind and consequently there exists a large variety of water lifting equipment (fig. 5.1). Here we shall limit ourselves to pumps that can be driven by wind rotors, with a focus on the reciprocating piston pump.

Broadly, pumps can be divided into three types, displacement, impulse and other pumps, each with a number of designs:

1. **Displacement pumps**
   - piston (plunger)
   - screw (Archimedean)
   - gear
   - worm
   - vane
   - roller
   - membrane
   - chain or ladder

2. **Impulse pumps**
   - centrifugal
   - axial

3. **Other pumps**
   - air-lift
   - ejector
   - hydraulic ram
   - siphon

Some advantages and disadvantages of different pumps are listed in fig. 5.2.
Fig. 5.1 Traditional water lifting devices. (Taken from: H. Stam, Adaptation of windmill designs, with special regard to the needs of the less industrialized areas, U.N. Conference on new sources of energy, Rome, Vol. 7, 1961, P. 347-357).
Advantages and Disadvantages of Various Types of Pumps

(Source: U.S. Dept. of Agriculture)

<table>
<thead>
<tr>
<th>Type</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plunger Type</td>
<td>Positive action (force)</td>
<td>Discharge pulsates</td>
</tr>
<tr>
<td></td>
<td>Wide range of speed</td>
<td>Subject to vibration</td>
</tr>
<tr>
<td></td>
<td>Efficient over wide range of capacity</td>
<td>Deep-well type must be set directly over well</td>
</tr>
<tr>
<td></td>
<td>Simple construction</td>
<td>Sometimes noisy</td>
</tr>
<tr>
<td></td>
<td>Suitable for hand or power operation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>May be used on almost any depth of well</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Discharge relatively constant regardless of head</td>
<td></td>
</tr>
<tr>
<td>Turbine Type</td>
<td>Simple design</td>
<td>Must have very close clearance</td>
</tr>
<tr>
<td></td>
<td>Discharge steady</td>
<td>Subject to abrasion damage</td>
</tr>
<tr>
<td></td>
<td>Suitable for direct connection to electric motor</td>
<td>Not suitable for hand operation</td>
</tr>
<tr>
<td></td>
<td>Practically vibrationless</td>
<td>Speed must be relatively constant</td>
</tr>
<tr>
<td></td>
<td>Quiet operation</td>
<td>Must be set down near or in water in deep well</td>
</tr>
<tr>
<td></td>
<td>May be either horizontal or vertical</td>
<td>Requires relatively large-bore well</td>
</tr>
<tr>
<td>Centrifugal Type</td>
<td>Simple design</td>
<td>Low efficiency in low capacities</td>
</tr>
<tr>
<td></td>
<td>Quiet operation</td>
<td>Low suction-lift (6 to 8 ft.)</td>
</tr>
<tr>
<td></td>
<td>Steady discharge</td>
<td>Must be set down near or in water in deep well</td>
</tr>
<tr>
<td></td>
<td>Efficient when pumping large volumes of water</td>
<td>Requires relatively large-bore well</td>
</tr>
<tr>
<td></td>
<td>Suitable for direct connection to electric motor</td>
<td>Not suitable for hand operation</td>
</tr>
<tr>
<td></td>
<td>Motor or for belt drive</td>
<td>Discharge decreases somewhat as discharge pressure increases</td>
</tr>
<tr>
<td></td>
<td>May be either horizontal or vertical</td>
<td></td>
</tr>
<tr>
<td>Rotary Type</td>
<td>Positive action</td>
<td>Subject to abrasion</td>
</tr>
<tr>
<td></td>
<td>Occupies little space</td>
<td>Likely to get noisy</td>
</tr>
<tr>
<td></td>
<td>Wide range of speed</td>
<td>Not satisfactory for deep wells</td>
</tr>
<tr>
<td></td>
<td>Steady discharge</td>
<td></td>
</tr>
<tr>
<td>Ejector Type</td>
<td>Simple construction</td>
<td>Jet nozzle subject to abrasion and clogging</td>
</tr>
<tr>
<td></td>
<td>Suitable either for deep or shallow wells</td>
<td>Limited to wells 120 feet or less in depth</td>
</tr>
<tr>
<td></td>
<td>Need not be set directly over well</td>
<td>Discharge decreases somewhat as discharge pressure increases</td>
</tr>
<tr>
<td></td>
<td>Quiet operation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Especially suitable for use with pressure system</td>
<td></td>
</tr>
<tr>
<td>Chain Type</td>
<td>Simple</td>
<td>Inefficient</td>
</tr>
<tr>
<td></td>
<td>Easily installed</td>
<td>Limited to shallow wells</td>
</tr>
<tr>
<td></td>
<td>Self-priming</td>
<td>Likely to be unsanitary</td>
</tr>
<tr>
<td>Hydraulic Ram</td>
<td>Simple design</td>
<td>Wastes water</td>
</tr>
<tr>
<td></td>
<td>Low cost</td>
<td>Likely to be noisy</td>
</tr>
<tr>
<td></td>
<td>Uses water for power</td>
<td>Not satisfactory for intermittent operation</td>
</tr>
<tr>
<td></td>
<td>Requires little attention</td>
<td></td>
</tr>
<tr>
<td>Siphon</td>
<td>Low cost</td>
<td>Limited to moving water to lower levels</td>
</tr>
<tr>
<td></td>
<td>Requires no mechanical or hand power except for starting</td>
<td>Requires absolutely airtight pipes</td>
</tr>
</tbody>
</table>

Fig. 5.2 Advantages and disadvantages of various types of pumps.

5.2 Piston pumps

A reciprocating piston pump basically consists of a piston, two valves and a suction and a delivery pipe. Sometimes airchambers are utilized to smooth the flow and to reduce shock forces. In the traditional piston pump the upper valve is mainly situated in the piston; the lower valve is called the foot valve (fig. 5.3). When piston and upper valve are separated one often employs the name plunger pump.

The operation principle of the reciprocating piston pump is simple: if the piston moves downward the upper valve opens, the foot valve closes; i.e. the flow is zero, and the piston moves freely through the watercolumn. As soon as the piston moves upward the upper valve will close, the foot valve opens and water is being lifted (above the piston) and sucked (below the piston, if the pump is above the water level) until the piston moves downward again. The result is a pulsating sinusoidal water flow, like an AC current after passing a rectifier (fig. 5.3). Apart from this so-called single-acting pump there are also double-acting pumps with two pistons moving in opposite directions, thus delivering water during the gaps in the flow of a single-acting pump. Here we shall consider only the single-acting pump, because the force required during the downward stroke requires precautions against buckling of the pump rod.

NOTE: In this section volumes appear for the first time. They are indicated with $V$ to avoid confusion with the velocity $V$.

5.2.1 Behaviour of ideal pumps

We shall now describe the behaviour of the ideal pump at low speed, i.e. with accelerations small compared to the acceleration of gravity and neglecting friction forces and dynamic forces.

The force on the piston is equal to the weight of the water column acting upon it, i.e. from the water level until the outlet (Note: also when the pump is below the water level).
Fig. 5.3 Schematic drawing of a reciprocating piston pump connected to a wind rotor.
We assume \( H \) to be the static head, but later on the extra head required to cover the losses has to be added.

This force \( F_p \) is transmitted to the crank of the wind rotor by the pump rod and exerts a torque on the rotor shaft via the crank arm \( r \), equal to half the stroke \( s \). The result is that the ideal torque is sinusoidal during the upward stroke and zero during the downward stroke. In formula:

\[
Q_{id} = \rho \omega g H \frac{\pi}{4} \frac{D^2}{p} \frac{1}{2} s \sin \omega t \text{ for } 0 < \omega t < \pi
\]

\[
Q_{id} = 0 \text{ for } \pi < \omega t < 2\pi
\]

Integrating this instantaneous torque over a full circle, gives, with the help of:

\[
\frac{1}{2\pi} \int_0^\pi \sin \omega t \, d\omega t = \frac{1}{\pi}
\]

the expression for the average torque:

\[
\overline{Q}_{id} = \frac{1}{\pi} \rho \omega g H \frac{\pi}{4} \frac{D^2}{p} \frac{1}{2} s
\]

or \( \overline{Q}_{id} = \frac{1}{2\pi} \rho \omega g H V_s \)

with \( V_s \) being the stroke volume. Note that the average torque is independent of the speed.

The average ideal power required is equal to:

\[
\overline{P}_{id} = \overline{Q}_{id} \cdot \Omega = \frac{1}{2\pi} \rho \omega g H V_s
\]

or \( \overline{P}_{id} = q \rho \omega g H \)
with \( q \) representing the average flow. Note that this is the nett hydraulic power to lift \( q \) m\(^3\)/s over a head of \( H \) meter.

5.2.2 Practical behaviour of pumps

The ideal pump of section 5.2.1 required a mechanical power equal to the nett power to lift the water, i.e. an efficiency of 100%. In reality the mechanical power required is higher than the nett water lifting power because of mechanical losses, due to friction between piston and cylinder, and hydraulic losses, due to flow friction losses in the valves mainly.

The mechanical efficiency is defined as:

\[
\eta_{\text{mech}} = \frac{P_{\text{hydr}}}{P_{\text{mech}}} \tag{5.6}
\]

in which \( P_{\text{hydr}} \) : nett power to lift the water \( (qp \omega gH) \)

\( P_{\text{mech}} \) : mechanical power driving the pump

(in our case the power from the rotor)

The volumetric efficiency arises because the actual output is usually less than the product of stroke volume, and speed.

Its definition is:

\[
\eta_{\text{vol}} = \frac{q}{V_s \frac{\Omega}{2\pi}} \tag{5.7}
\]

with \( V_s \) being the stroke volume \( \frac{\pi D^2}{4} \).

Please note that these volumetric losses have little to do with the hydraulic losses due to flow friction, so the volumetric efficiency in principle does not appear in power calculations.
Fig. 5.4 Power-speed and flow-speed relations of a piston pump, indicating the mechanical and volumetric losses.

For illustration purposes the measurement results of an actual pump are shown in fig. 5.5. The ideal torque is the torque of equation (5.4).

A relation between the two efficiencies can be found via the ideal and the real (mechanical) torques required. This relation will be used in section 6.4. Relation 5.6 can be written as follows:

\[
\eta_{\text{mech}} = \frac{q \rho w g H}{\bar{Q}_{\text{mech}} \Omega} = \frac{\eta_{\text{vol}} \bar{V} / s \frac{\Omega}{2\pi} \rho w g H}{\bar{Q}_{\text{mech}} \Omega} \quad (5.8a)
\]

with the equation (5.4) for the ideal torque this becomes:

\[
\eta_{\text{mech}} = \frac{\eta_{\text{vol}} \bar{Q}_{\text{id}}}{\bar{Q}_{\text{mech}}} \quad (5.8b)
\]
Fig. 5.5 Characteristics of the SWD Tunesia pump.
- Diameter: 0.141 m
- Stroke: 0.080 m
- Valve clearance: 0.005 m
- Head (used here): 11.4 m
5.3 **Acceleration effects**

The behaviour of the reciprocating piston pump, as described in section 5.2, represents the low-speed behaviour of the pump. At higher speeds, say 1 to 2 strokes per second and above, acceleration effects cannot be neglected anymore. With long suction pipes there is the risk of cavitation and the high acceleration will delay the valve closure considerably, leading to unexpected high shock forces. At very high accelerations the piston pump will behave more as an impulse pump, leading to increased volumetric efficiency (even above 100%) but a sharply reduced mechanical efficiency. Before describing these effects in detail, we shall present a simple mathematical model for the piston pump.

A piston pump, directly driven via a crank, exhibits a nearly sinusoidal displacement of the piston as a function of the angular rotation of the driving shaft (fig. 5, 6). Taking the displacement $z$ and the time $t$ as zero when the piston is at its bottom position (we assume a vertical movement of the piston), then the position of the piston $z_p$ as a function of angular rotation is given by:

$$z_p = \frac{1}{2}s - \frac{1}{2}s \cos \Omega t$$  \hspace{1cm} (5.9)

Two subsequent differentiations give us velocity and acceleration ($\Omega$ is constant):

$$v_p = \frac{1}{2}\Omega s \sin \Omega t$$  \hspace{1cm} (5.10)

$$a_p = \frac{1}{2}\Omega^2 s \cos \Omega t$$  \hspace{1cm} (5.11)

Together with the displacement these function are shown in fig. 5.6 on the next page. To give an idea about their actual values, we used a stroke of 0.2 m and a speed of 1 revolution per second as an example.
Fig. 5.6
The displacement, velocity and acceleration of an ideal piston pump with following characteristics:

\[ s = 0.2 \text{ m} \]
\[ \Omega = 2\pi \text{ rad/s} \]
For the study of acceleration effects we are often interested in
the ratio between the actual maximum acceleration and the
acceleration of gravity. We introduce their ratio as the
acceleration coefficient \( C_a \) [22]:

\[
C_a = \frac{a_{\text{Pmax}}}{g} = \frac{b\Omega^2\rho}{g}
\]  (5.12)

In our example \( C_a = 0.4 \). Values of \( C_a \) above 1 imply a
compressive force on the pump rod during the downward stroke (with
the danger of buckling) because the pump rod is forced to
accelerate faster than gravity would do.

Because extra upward forces are caused by the friction of the cup
leather of the piston and the pressure difference over the piston
valve one usually limits \( C_a \) to values below 1, with \( C_a = 0.5 \) as
a reasonable value. This is accomplished by limiting the stroke of
the pump, because the maximum speed is usually already determined
by other factors. In commercial windmills often a gear box is
found, instead of a crank, thereby reducing the pump speed and
enabling the stroke to become proportionally longer.

Even with values of \( C_a \) lower than 1, acceleration problems might
occur. This is caused by the possibility of cavitation below the
piston which occurs when the local pressure falls below the vapour
pressure. The introduction of an air chamber, also necessary for a
more regular flow and the reduction of shock forces, can prevent
cavitation. We shall describe the behaviour of the two idealised
pumps given in fig. 5.7 and fig. 5.8, in order to predict the
cavitation.

If the perfectly fitting piston in fig. 5.7 suddenly moves upward,
a vacuum will be created between the piston and the top of the
water column. Assuming that \( H < 10 \text{ m} \), the surrounding atmospheric
pressure on the water below will push the water column upward to
reach the 10 m level in the end if the piston would move that far.
We can calculate the acceleration of the water column \( a_w \) by
calculating force and mass, assuming that its diameter is equal to
the diameter of the piston \( D_p \).
Idealised piston pump
without air chamber.

Idealised piston pump
with suction air chamber.

\[
\text{Force} = P_{\text{atm}} \left( \frac{\pi}{4} D^2 \right) - \rho g H \left( \frac{\pi}{4} D^2 \right) \quad (5.13)
\]

or
\[
\text{Force} = \rho g (10-H) \left( \frac{\pi}{4} D^2 \right) \quad \text{(at sea level)} \quad (5.14)
\]

and
\[
\text{Mass} = \rho L \left( \frac{\pi}{4} D^2 \right) \quad (5.15)
\]

So the acceleration \( a_w \) becomes:
\[
a_w = \frac{10-H}{L} \cdot g \quad (5.16)
\]

This shows that even in the simple case of a 5 m lift and a 10 m long suction pipe the acceleration cannot be higher than 0.5 g or else cavitation will occur. Heavy shocks will be the result each time the accelerated water column hits the piston, which is slowing down during its upward movement. It will be clear that we must ensure that \( a_w > a_{\text{max}} \) in order to avoid cavitation. In other words:
\[
\frac{10-H}{L} g > \frac{1}{4} \Omega^2_s \quad (5.17)
\]
The best way of avoiding cavitation is to avoid high suction heads, or, if this is not possible, to reduce the length \( L \) of the water column concerned, by introducing an air chamber (fig. 5.8). In this case the water column with height \( h \) and the water in the air chamber add up to a total column with length \( l \) and mass \( \rho l \frac{\pi}{4} D^2 \). This mass is driven by the (average) pressure in the air-chamber roughly equal to \((10-H_{ac})\) m water column, minus the small head \((H-H_{ac})\) of the water in the air chamber. The result is that the water column with equivalent length \( l \) is driven by a pressure equal to \((10-H)\) m water column and the acceleration becomes:

\[
a_w = \frac{10-H}{l} \times g
\]  

(5.18)

and we see that we can increase the acceleration by reducing \( l \) to fulfill the non-cavitation condition (5.17).

So far we have concentrated on the acceleration of the suction water column, but obviously similar effects play a role in the delivery line. The water column above the piston is accelerated in the beginning of the upward stroke with the acceleration \( a_p \) of the piston, whatever its value, because the piston valve is closed. When the piston begins to decelerate, however, the water column will not be able to follow if the deceleration is larger than \( g \), because the water column then becomes simply a free body, moving upward and at the same time being decelerated by gravity. This does not happen when a pressure air chamber is installed, because in that case the pressure in the air chamber will ensure a quick recharge of water above the piston. We shall continue to discuss the situation without pressure air chamber.

When the deceleration of the piston exceeds \( g \) during the last part of the upward stroke, the water column will continue to move upward on its own, now decelerated by \( g \) only. This means that the piston valve has to open and that an extra amount of water passes the piston valve. In other words, the pump will displace more water than its stroke volume and its volumetric "efficiency" can become higher than 100%. This is illustrated in fig. 5.9.
Fig. 5.9

The displacement velocity and acceleration of an ideal piston pump. The rotational speed is $\Omega = 4\pi$ rad/s, i.e. twice the speed as given in fig. 5.6.
The position angle $\Omega t_1$ at which an acceleration of $-g$ is reached is found with:

$$-g = \frac{1}{2}a_s \cos \Omega t_1$$

so

$$\cos \Omega t_1 = \frac{-g}{\frac{1}{2}a_s} = - \frac{1}{\frac{1}{2}a_s}$$

In the example of fig. 5.9 with $\Omega = 4\pi$ rad/s and $s = 0.2$ m, the acceleration coefficient becomes $a_s = 1.61$ and the position angle becomes $\Omega t_1 = 2.241$ rad = 128.4°. From this moment on the water column above the piston is "launched" and moves upward, although its velocity is linearly decreasing with a velocity $-g(t-t_1)$.

When the velocity decreases to zero the foot valve will close.

The position angle $\Omega t_2$ at which the water speed becomes zero, is found by realizing that the speed at $t_1$, when the piston valve opens, can be calculated with (5.10), (5.20) and $\sin \Omega t = \sqrt{1 - \cos^2 \Omega t}$:

$$v_w(t_1) = v_p(t_1) = \frac{1}{2} a_s \sqrt{1 - \frac{1}{\frac{1}{2}a_s}}$$

Now the linearly decreasing speed is given by

$$v_w = v_w(t_1) - g(t-t_1)$$

or

$$v_w = v_w(t_1) - \frac{g}{\Omega} (\Omega t - \Omega t_1)$$

and with (5.21) and (5.20) this becomes:

$$v_w = \frac{1}{2}a_s \sqrt{1 - \frac{1}{\frac{1}{2}a_s}} - \frac{g}{\Omega} (\Omega t - \arccos(-\frac{1}{\frac{1}{2}a_s}))$$
Zero water speed is reached at the position $\Omega t_2$, which can be found by substituting $V_w = 0$ in (5.23):

$$
\Omega t_2 = \sqrt{(C_a^2 - 1)} + \arccos \left( -\frac{1}{C_a} \right)
$$

This expression is valid for $\Omega t_2 < 2\pi$, otherwise a new stroke has already begun (see below).

In our example we find $V_w(t_1) = 0.985 \text{ m/s}$ and $\Omega t_2 = 3.502 \text{ rad} = 200.7^\circ$.

The extra amount of water displaced can be found by calculating the extra displacement $z_w - s$ (fig. S.9) of the water column at the moment $t = t_2$. The position of the water column at $t = t_2$ is given by:

$$
z_w(t_2) = z_w(t_1) + V_w(t_1) \ast (t_2 - t_1) - \frac{1}{2} g (t_2 - t_1)^2
$$

Substitution of (5.20) and (5.21) in (5.25) yields:

$$
z_w(t_2) - s = \frac{1}{2} s \left(1 - \cos \Omega t_1 \right) + \frac{1}{2} s \sqrt{\left(1 - \frac{1}{C_a^2}\right) \ast \frac{1}{\Omega} \sqrt{\left(C_a^2 - 1\right) - g \frac{C_a^2 - 1}{\Omega^2}}} - s
$$

$$
z_w(t_2) - s = \frac{1}{2} s \left(1 + \frac{1}{C_a}\right) + \frac{1}{2} s \frac{C_a^2 - 1}{C_a} - \frac{1}{4} s \frac{C_a^2 - 1}{C_a}
$$

$$
z_w(t_2) - s = s \left[ -\frac{1}{2} + \frac{1}{4C_a} + \frac{C_a}{4} \right] \text{ for } \Omega t_2 < 2\pi.
$$

Substitution of $C_a = 1.61$ of our example leads to $z_w(t_2) - s = s \ast 0.0575$, so in the ideal case the volumetric efficiency is increased to $(1 + 0.0575) \ast 100\% = 106\%$. 
In cases where \( \Omega t_2 > 2\pi \) we may conclude that the water speed never becomes zero, because at \( \Omega t = 2\pi \) the next upward stroke begins. In other words: the piston pump has become an impulse pump (in this case sometimes called inertia pump), pushing the water column upwards by rapid impulses instead of gradually lifting it (fig. 5.10). The water velocity is always positive now, so we could even remove the footvalve of the pump. The value of \( C_a \) to reach this condition is found with expression (5.24):

\[
\sqrt{(C_a^2 - 1)} + \arccos \left(1 - \frac{1}{C_a}\right) = 2\pi
\]

Via iteration one finds: \( C_a = 4.6 \) (4.6033389 to be exact). In our example with \( s = 0.2 \) m the speed must be increased to \( \Omega = 21.25 \) rad/sec = 3.38 r.p.s. to reach this state. The ideal volumetric efficiency can be found with (5.26) and has increased to 170.5%.

In the situations with \( C_a > 4.6 \) the intersection of the linearly decreasing water speed (5.22) and the increasing piston speed must be found, in order to find \( \Omega t_2 \) and the corresponding volumetric efficiency:

\[
W(t_1) - g(t_2 - t_1) = \frac{1}{2}gs \sin(\Omega t_2)
\]

\[
\frac{1}{2}gs \sqrt{1 - \frac{1}{C_a^2}} - \frac{2}{\Omega}(\Omega t_2 - \Omega t_1) = \frac{1}{2}gs \sin(\Omega t_2)
\]

\[
\sqrt{1 - \frac{1}{C_a^2}} \cdot \frac{1}{C_a} \cdot \{ \Omega t_2 - \arccos \left(-\frac{1}{C_a}\right) \} = \sin(\Omega t_2)
\]

(5.27)
The solution of (5.27) cannot be found analytically but is easy to handle with a calculator. For very high values of $C_a$ one approaches $\Omega t = 2\pi + \frac{1}{2}\pi$, i.e. at the moment of maximum piston speed during the next stroke. The water speed now approaches a constant speed equal to the maximum piston speed, so this (purely theoretical) pump could achieve a volumetric efficiency of $\pi$. (Remember that the average flow of a single acting piston pump is equal to $\frac{1}{\pi}$ times the maximum flow.)

![Diagram of piston displacement pump transition to impulse pump](image)

**Fig. 5.10** The transition of the piston displacement pump to an impulse pump.

When we introduce a pressure air chamber in the delivery line, the situation becomes quite different. The reader is asked to analyse this situation by him (or her) self.
5.4. Valve behaviour*

The two acceleration effects of section 5.3 were described for an ideal pump, i.e. with a perfectly fitting piston and with valves closing immediately when the flow velocity becomes zero. In practice this is not the case and at higher speeds the valves tend to close later than they should do, because of their inertia. This means that the flow has already reversed before the valve can close, and when it actually closes a water hammer effect occurs: large shock forces are experienced by the pump rod.

In this section we shall describe the behaviour of disk-shaped free-floating valves in reciprocating piston pumps (see fig. 5.11).

Fig. 5.11 Schematic drawing of the foot valve of a reciprocating piston pump.

*) The detailed description of valve behaviour has been given by J. Snoeij in his "Dynamic behaviour of free valves in piston pumps" (in Dutch), Internal report R 430 S, Eindhoven University of Technology, the Netherlands, 1980.
There are a number of forces acting on the valve, acting downward (negative sign) or upward.

- **Weight of the valve**: \(-m_v g\)  
  \((m_v: mass\ of\ valve)\)

- **Buoyancy force**: \(\rho_V g V_v\)  
  \((V_v: volume\ valve)\)

- **Stationary drag force**: \(F_{st}\)

- **Instationary drag force**: \(F_{inst}\)

- **Static force (when closed)**: \(-F_{cl}\)

Newton's law now can be written as:

\[
m_v a_v = -m_v g + \rho_V g V_v + F_{st} + F_{inst} - F_{cl}\]  

\((5.28)\)

Of which the last three terms will be discussed below. Note that we neglect the friction between the valve and its guidance.

Conservation of mass requires that the mass flow below the piston is equal to the mass entering the valve:

- **Mass flow below piston**: \(\rho_w A_p V_p\)

- **Mass flow in gap below valve**: \(\rho_w \pi D_z V_v g\)  
  \((D_z: diameter\ valve)\)

- **Mass flow below moving valve**: \(\rho_w A_v V_v\)  
  \((A_v: area\ valve)\)

- **Mass flow due to expanding pump cylinder with closed valves**: \(\rho_w \frac{V_{cyl}}{E} \frac{dp}{dt}\)

The last term is the result of Hooke's law for fluids, with compressibility modulus \(E\), pump cylinder volume \(V_{cyl}\) and pressure difference \(p\). This term plays a role only for very small values of \(z_v\) and is important in opening the valve. The total conservation law now is given by:
The three terms in (5.28) are described as follows. Usually one describes the static drag force on a flat valve is described with:

$$F_{\text{st}} = \frac{1}{2} \rho_w C_w \frac{A_v^2}{\mu}$$

(5.30)

With $\mu$ being a constant of which the value depends on the type of valve and the ratio between the gap area and the valve entrance area. Often $\mu$ is taken to be equal to 0.8.

Much less is known about the instationary force on a valve accelerating in a fluid. One can imagine that a given amount of water above the valve has to be accelerated, causing the extra force. In the case of a sphere moving in a fluid, this amount of added mass is assumed to be equal to the mass of fluid with a volume equal to half the volume of the sphere. The acceleration of the fluid above the valve is dictated by the acceleration $a_p$ of the piston (multiplied with the ratio $A_p/A_v$) minus the acceleration $a_v$ of the valve. The result is:

$$F_{\text{inst}} = \rho_w \nabla \frac{A_v}{\left( \frac{b}{D_v} \right) A_v (a_p - a_v)}$$

(5.31)

when we take the added volume as the volume of half a sphere with diameter $D_v$ then:

$$\nabla \text{added} = \frac{\pi}{12} D_v^3$$

(5.32)

In the situation of a perfectly closed valve the pressure $p_{av}$ above the valve acts on area $A_v$, but the pressure $p_{bv}$ below the valve acts only on the area $A_{vo}$ of the valve opening:

$$F_{\text{cl}} = -p_{bv} A_{vo} + p_{av} A_v$$

(5.33)
The equilibrium of forces for the closed valve just before opening becomes:

\[-p_{bv} A_{vo} + p_{av} A_v - mg + \rho_g g v^2 = 0\]  \hspace{1cm} (5.34)

Substitution of the different terms in equation (5.28) leads to the following non-linear differential equation:

\[
m \frac{d^2 z}{dt^2} + (m - \rho_w v)g - (\text{sign } F_{st}) \frac{\rho_w A_v (\rho_v - \rho_p) - \frac{dz}{dt} + \frac{\Delta V}{E} \frac{dp}{dt}}{2 \mu^2 \pi^2 D_v^2 z^2} \\
- \rho_v \dot{V} \text{ added} \left( \frac{A_p}{A_v} \dot{a} - \frac{d^2 z}{dt^2} \right) + F_{cl} = 0 \]  \hspace{1cm} (5.35)

This equation cannot be solved analytically, so numerical solution is necessary. We shall not bother here with how to solve the equation, but instead give a few results for a typical situation and compare these results with actual laboratory measurements.

The characteristics of the typical situation are as follows:

- Diameter piston \( D_p = 0.14 \) m
- Diameter valve \( D_v = 0.12 \) m
- Thickness valve \( t_v = 0.014 \) m
- Density valve \( \rho_v = 3.78 \times 10^3 \) kg/m\(^3\)
- Stroke \( s = 0.08 \) m
- Volume added mass \( \dot{V} \text{ added} = \frac{\pi}{12} D_v^3 m^3 \)
- Maximum gap height \( z_{\text{max}} = 0.006 \) m
The results are shown in the following figures 5.12, 5.13 and 5.14.

Fig. 5.12 The influence of the maximum gap height $z_m$ on the closure angle of the foot valve.
Fig. 5.13 Angular positions of piston valve (upper graph) and foot valve (lower graph) at closure of the respective valves. The theoretical (drawn) curves are for zero added water volume and for an added volume equal to $\frac{\pi}{12} D_p^3$.
Fig. 5.14 The influence of the density of the valve material on the closure angles of the piston valve (upper graph) and the foot valve (lower graph).
Note:
A rather crude approximation may help in the physical understanding of the closure behaviour of the piston valve (fig. 5.14). At high rotational speeds we could assume that, at the lowest position of the piston, the valve is more or less standing still in space. Then the piston is moving upward until it meets the valve and carries it upward. In this simple approximation we only need to calculate the angle \( \Omega t \) to move the piston from its lowest position to the maximum gap height \( z_{\text{max}} \). With formula (5.9) one sees that

\[
\hat{z}_{\text{max}} = \frac{1}{2} \eta \cos \Omega t.
\]

\[
\Omega t = \arccos \left(1 - \frac{\hat{z}_{\text{max}}}{\frac{1}{2} \eta} \right)
\]

With \( z_{\text{max}} = 0.006 \) m and \( s = 0.08 \) m this angle becomes:

\( \Omega t = 32^\circ \) or \( 392^\circ \).

We can see in fig. 5.14 that lighter valves than the ones indicated \((\rho_v < 2 \times 10^3 \text{ kg/m}^3)\) could possibly approach values around \( 390^\circ \) at high speeds.
5.5 **Air chambers**

5.5.1 General

The shock forces experienced by the piston at the moments the valves close, strongly depend on the mass of water above and under the valves. The installation of air chambers near the valves softens the influence of all mass of water beyond the air chambers and can greatly reduce the forces on the piston, and consequently reduce the forces on the pump rod and the bearings.

The mass of water in the pipe beyond an air chamber, however, acts as a mass-spring system together with the compressible air in the air chamber. This means that unwanted resonances can occur that can cause as much damage as the forces that the air chambers were supposed to prevent.

In this section we shall describe the behaviour of (ideal) air chambers and present some guidelines for their design. In fig. 5.15 a schematic representation of an air chamber is given.

![Schematic representation of the suction air chamber for a reciprocating piston pump.](image)

Fig. 5.15 Schematic representation of the suction air chamber for a reciprocating piston pump.
5.5.2 Resonance frequency

The resonance frequency of the ideal airchamber (i.e. without damping) can be calculated since we know that the resonance angular frequency $\Omega_0$ of a system with a mass $m$ and a spring constant $k$ is given by:

$$\Omega_0 = \sqrt{\frac{k}{m}} \text{ (rad/s)} \quad (5.36)$$

The mass involved in the oscillations is the mass of all water below the air chamber (in fig. 5.15). For pressure air chambers this is obviously the mass of all water above the airchamber. When we assume that the area of air chamber and suction pipe are equal ($A_a = A_s$, see fig. 5.15), we find

$$m = \rho A_s L \quad (5.37)$$

The spring constant $k$ is found with the following reasoning. A water displacement $dL$ in the air chamber causes a volume variation $-dV_a$ in the air volume of the air chamber, resulting in a pressure change $dp_a$ of the pressure $p_a$ in the air chamber. The volume variation is equal to:

$$-dV_a = A_s dL \quad (5.38)$$

and the pressure variation causes a force $dF$ by the air on the water:

$$dF = A_s dp_a \quad (5.39)$$

Because the spring constant is defined as the ratio of $dF$ and $dL$ we find:

$$k = \frac{dF}{dL} = \frac{-A_s dp_a}{\frac{dV_a}{A_p}} = -A_s^2 \frac{dp_a}{dV_a} \quad (5.40)$$
Substitution of (5.40) and (5.37) in (5.36) yields the resonance angular frequency:

\[ \omega_o = \sqrt{\frac{-A^2}{s \frac{dV}{dV}}} \]

(5.41)

The derivative \( \frac{dp}{dV} \) is found with the gas law of Boyle-Gay Lussac for an ideal gas:

\[ p \gamma = \text{constant} \]

(5.42)

The exponent \( \gamma \) is equal to 1.0 for isothermal processes and equal to 1.4 for adiabatic processes. In most cases the compression and expansion of the air in the air chamber will take place so rapidly that it can be regarded as an adiabatic process, so we choose \( \gamma = 1.4 \). Taking now the total derivative of the gas law (5.42) we find:

\[ p \gamma \frac{dV}{dV} + \gamma \frac{dp}{dV} = 0 \]

(5.43)

and so

\[ \frac{dp}{dV} = -\gamma \frac{p}{V} \]

(5.44)

Using the result in the expression (5.41) for the resonance angular frequency we find:

\[ \omega_o = \sqrt{\frac{\gamma A_s p_a}{\rho_w a L}} \]

(5.45)
As an example we shall calculate the resonance frequency of a piston pump with a pressure air chamber under the following conditions:

\[ A_s = 0.004 \text{ m}^2 \quad L = 15 \text{ m} \]

\[ p_a = 2 \times 10^5 \text{ N/m}^2 \quad \gamma = 1.4 \]

\[ V_a = 0.011 \text{ m}^3 \quad \rho_w = 1000 \text{ kg/m}^3 \]

The result is: \( \Omega_0 = 2.6 \text{ rad/s (or 0.41 Hz)} \)

So far we have assumed that the suction pipe and the air chamber have the same diameter. In case the diameters are different (fig. 5.16) the formulas change somewhat as is shown in ref. 23. If we assume that the piping arrangement consists of a number of pipes each with length \( L_1, L_2, L_3 \), etc. and areas respectively \( A_{s1}, A_{s2}, A_{s3} \), etc. then the following formula holds

\( \Omega_0 = \sqrt{\left( \frac{\gamma p_a}{\rho_w V_a \left( \frac{L}{A_a} + \frac{L_1}{A_{s1}} + \frac{L_2}{A_{s2}} + \ldots \right)} \right)} \)  \( (5.46) \)
Fig. 5.16 An air chamber connected to a pipe with different diameters.

The practical use of the resonance calculation lies in the fact that one can calculate the minimum volume required for the air volume of the air chamber in order to make sure that the resonance frequency remains below the normal operating frequencies of the pump. In that case the air chamber is going through resonance during the one or two strokes to arrive at the operating speed and that is relatively harmless. As a guideline one usually chooses a resonance frequency which is 1.5 times smaller than the minimum operating frequency of the pump. This is based on the fact that the amplitude of an ideal oscillator is sharply reduced above its resonance frequency and at 1.5 times the resonance frequency its amplitude has become smaller than the amplitude of the oscillating force that drives the oscillator (see section 5.5.4.).
5.5.3 Volume variations

We assume that an ideal air chamber is coupled to a single-acting piston pump. Ideal in this respect means that the outgoing flow is perfectly constant (fig. 5.17).

\[ q = A_p \frac{\Omega}{2} \frac{s}{\pi} \sin \omega t \text{ (m}^3/\text{s)} \text{ for } 0<\omega t<\pi \]

\[ q_{in} = 0 \quad \text{for } \pi<\omega t<2\pi \]

(5.47)

Fig. 5.17 The in and outgoing flow of an ideal air chamber coupled to a single-acting piston pump.

If the pump has a piston diameter \( A_p \), a stroke \( s \) and a rotational speed \( \Omega \), then the incoming flow to the air chamber can be described as follows:

The outgoing flow is assumed to be constant and must be equal to the average of the incoming flow:

\[ q_{out} = A_p \frac{\Omega}{2} \frac{s}{\pi} \frac{1}{2\pi} \int_0^{2\pi} \sin \omega t \, d\omega t \]
resulting in:

\[ q_{\text{out}} = A_p \Omega \frac{h}{\pi} \]  

(5.48)

Comparing (5.48) with (5.47) we see that only for the moment at which \( \sin \Omega t = \frac{1}{\pi} \) the ingoing and outgoing flows are equal. For all other moments the flows are different, or, in other words, the air chamber has to absorb or supply water. The two moments, or better, position angles, concerned can be found with:

\[ \Omega t = \arcsin \frac{1}{\pi} + \Omega t_1 = 0.324 \text{ rad or 18.56°} \]  

(5.49)

\[ \Omega t_2 = 2.818 \text{ rad or 161.44°} \]

At \( t_1 \) the water volume of the air chamber is at its minimum and at \( t_2 \) the water volume is maximal. In order to find the volume variations in the ideal air chamber we have to integrate \( (q_{\text{in}} - q_{\text{out}}) \) with respect to \( t \) (fig. 5.18).

For \( 0 < \Omega t < \pi \) we can write:

\[ \int (q_{\text{in}} - q_{\text{out}}) \, dt = A_p \frac{h}{s} \int (\sin \Omega t - \frac{1}{\pi}) \, d\Omega t \]

\[ = A_p \frac{h}{s} (-\cos \Omega t - \frac{\Omega t}{\pi} + \text{constant}) \]

The boundary condition of zero volume at \( \Omega t = 0 \) leads to:

\[ \int (q_{\text{in}} - q_{\text{out}}) \, dt = A_p \frac{h}{s} (1 - \cos \Omega t - \frac{\Omega t}{\pi}) \]  

(5.50)

For \( \pi < \Omega t < 2\pi \) one finds:

\[ \int (q_{\text{in}} - q_{\text{out}}) \, dt = A_p \frac{h}{s} (2 - \frac{\Omega t}{\pi}) \]  

(5.51)

These two parts of the function are shown in the lower graph of fig. 5.18; note that the stroke volume is equal to \( V_s = A_p s \).
Fig. 5.18 Determination of volume variations of the ideal air chamber, i.e. with a constant outgoing flow.

The difference between the minimum and maximum volume can be found with (5.49) and (5.50):

$$V_{\text{max}} - V_{\text{min}} = \frac{1}{2} V_s \left(1 - \cos \Omega t_2 - \frac{\Omega t_2}{\pi} - 1 + \cos \Omega t_1 + \frac{\Omega t_1}{\pi}\right)$$

$$V_{\text{max}} - V_{\text{min}} = 0.551 V_s \quad (5.52)$$
5.5.4 Damped mass-spring systems

The general behaviour of a damped mass-spring system driven by a sinusoidal outside force is described by a well-known second-order differential equation:

\[ m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = F_o \sin \omega t \]  

or \[ \frac{d^2 x}{dt^2} + 2 \beta \Omega_o \frac{dx}{dt} + \Omega_o^2 x = \frac{F_o}{m} \sin \omega t \]

with:
- \( m \): mass
- \( c \): friction coefficient
- \( k \): spring constant
- \( \Omega \): outside angular frequency
- \( F_o \): amplitude of outside force
- \( \Omega_o = \sqrt{k/m} \): resonance angular frequency
- \( \beta = \frac{c}{2m\Omega_o} \): damping coefficient

The solution of (5.52) is given in many textbooks and consists of a steady-state response and a transient response, the latter with an amplitude which decreases with time:

\[ x(t) = \frac{F_o}{\sqrt{m^2 (\Omega_o^2 - \Omega^2)^2 + \frac{c^2}{4m^2 \Omega_o^2}}} \cos (\Omega t - \arctan \frac{c \Omega}{m (\Omega^2 - \Omega_o^2)} + \frac{c}{2m} t \cos \left\{ \Omega_o \sqrt{1 - \frac{c^2}{4m^2 \Omega_o^2}} t - \psi \right\} \]

with \( A \) and \( \psi \) depending on the initial conditions.
The dimensionless damping coefficient $\beta$ simplifies (5.54) into:

$$
x(t) = \frac{F_0}{m\Omega_0^2} \cos (\Omega t - \arctan \frac{2\beta \Omega_0}{\Omega_0^2}) + \sqrt{(1 - \frac{\Omega_0^2}{\Omega_0^2}) + 4\beta^2 \frac{\Omega_0^2}{\Omega_0^2}} \left(1 - \frac{\Omega_0^2}{\Omega_0^2}\right) + e^{-\beta \Omega_0 t} \cos \left\{ \left(\Omega_0 \sqrt{1 - \beta^2} - \psi \right) t \right\}
$$

(5.55)

The maximum value of the amplitude of the steady-state response is reached at the resonance angular frequency $\Omega_r$:

$$
\Omega_r = \Omega_0 \sqrt{1 - 2\beta^2}
$$

(5.56)

The variation of the relative amplitude ($F_0/m\Omega_0^2 = 1$) and phase of the steady-state response is shown in fig. 5.19.

Note that for $\frac{\Omega}{\Omega_0} > 1.5$ the relative amplitude is always below 1, regardless of the value of the damping coefficient $\beta$. 
Fig. 5.19 The relative amplitude (above) and the phase (below) of the steady-state response of a damped mass-spring system driven by an outside sinusoidal force. The relative amplitude is defined as the ratio of the amplitude of the mass-spring system and the amplitude of the driving force.
5.5.5 Dynamic behaviour of air chambers

The dynamic behaviour of a fluid in a pipe is governed by three factors: static pressure, friction losses and acceleration forces. For the pipe as shown in fig. 5.20 the following equation can be written down (see textbooks on fluid dynamics).

\[ p_1 - p_2 = \rho g H + f \frac{L}{D} \frac{1}{2} \rho v^2 + \rho L \frac{dv}{dt} \]  \hspace{1cm} (5.57)

static friction acceleration pressure

If we regard this pipe to be the delivery pipe of a piston pump with a pressure air chamber, then we arrive at the situation of fig. 5.21.
The incoming flow $q_w$ enters the system with a speed $V_w$ and is distributed between air chamber and delivery pipe in such a way that mass is conserved:

$$q_w = q_a + q_d \quad (5.58)$$

For the air chamber applies that the incoming flow $q_a$ is equal to the decrease in volume per unit of time (see 5.44).

$$q_a = -\frac{dV_a}{dt} = \frac{A_a}{\gamma p_a} \frac{dp_a}{dt} \quad (5.59)$$

Rewriting (5.57) in terms of $q_d$ gives:

$$p_d - p_o = \rho g H + f \frac{L_d}{D_d} \frac{1}{2} \frac{q_d^2}{A_d^2} + \frac{\rho L_d}{A_d^2} \frac{dq_d}{dt} \quad (5.60)$$
To simplify our calculations we assume that we may neglect the small pressure head, the friction losses and the acceleration losses in the air chamber and in the piece of pipe before the air chamber. The result is that the pressure in the air chamber is assumed to be equal to the pressure at the entrance of the delivery pipe:

\[ p_a = p_d \] (5.61)

Substitution of (5.59) (5.60) and (5.61) in (5.58) yields:

\[ q_w = \frac{V}{\gamma p_a} \left\{ f \frac{L_d}{D_d} \frac{1}{A_q^2} 2 q_d \frac{dq_d}{dt} + \frac{\rho L_d}{A_d} \frac{d^2 q_d}{dt^2} \right\} + q_d \] (5.62)

After rearranging:

\[ \frac{d^2 q_d}{dt^2} + f \frac{dq_d}{dt} + \frac{\gamma A_d p_a}{\rho L_d V_d} q_d = \frac{\gamma A_d p_a}{\rho L_d V_d} q_w \] (5.63)

In the third and fourth term we recognize the resonance frequency found in 5.5.2:

\[ \omega^2 = \frac{\gamma A_d p_a}{\rho L_d V_d} \]

so (5.63) can be written as:

\[ \frac{d^2 q_d}{dt^2} + f \frac{dq_d}{dt} + \omega_d^2 q_d = \omega_o^2 q_w \] (5.64)
We can transform this non-linear differential equation into a dimensionless equation. In section 5.2 we have seen that the average ideal flow is given by \((\Omega/2\pi) \cdot V_s\) so we can introduce a dimensionless flow \(\phi\) with:

\[
\phi = q \frac{2\pi}{\Omega \cdot V_s}
\]  

(5.65)

The corresponding dimensionless time \(T\) is given by:

\[
T = \Omega_o t
\]  

(5.66)

The resulting differential equation becomes:

\[
\frac{d^2\phi}{dt^2} + \frac{fV_s}{D \cdot A \cdot 2\pi} \cdot \phi + \frac{d\phi}{dT} + \phi = \phi_w
\]  

(5.67)

We see that we have found a sort of "damping coefficient" equal to

"damping coefficient":

\[
\frac{f \cdot V_s \cdot d}{D \cdot A \cdot 2\pi} = \frac{fV_s \cdot d}{\frac{1}{2} \cdot \pi^2 \cdot D^3}
\]  

(5.68)

if we linearize the differential equation for a moment.

This damping coefficient gives us a rough estimate to judge whether the damping will be large enough to reduce the fluctuation at resonance or not. As an example we shall use the data of the example in section 5.5.2 with a delivery pipe diameter of \(D_d = 0.071\) and assuming a friction coefficient \(f = 0.03\) (G.I. pipe). If this pipe is coupled to the SWD "Tunisia" pump with a stroke volume of \(V_s = 0.00125\) m³ (see fig. 5.5) then we find at resonance with \(\phi_d = 1\): damping coefficient = 0.02

In other words, the system nearly behaves like an undamped system, which will fluctuate dangerously when operated near resonance. Safe operation occurs for rotational frequencies higher than \(1.5\Omega_o\) as we have seen in section 5.5.4 and we conclude that the resonance frequency \(\Omega_r\) is only slightly smaller than \(\Omega_o\) because of the low value of the "damping coefficient" (see formula 5.56).
6. COUPLING OF PUMP AND WIND ROTOR

6.1 Description and example

If a pump is coupled to a wind rotor at a given wind speed \( V \) the rotor will turn at a speed such that the mechanical power of the rotor is equal to the mechanical power exerted by the pump. This working point can be found by the intersection of the rotor curve and the pump curve (fig. 6.1).

![Diagram of a graph showing the working point of a rotor-pump combination at a given wind speed \( V \).](image)

**Fig. 6.1** Working point of a rotor-pump combination at a given wind speed \( V \).

The actual flow of water lifted by the rotor-pump combination at the given wind speed is found by drawing the \( P_{\text{hydr}} \) curve (fig. 6.1), noting the power at the rotational speed of the working point and divide by \( \rho \cdot g \cdot H \).

To find the hydraulic output as a function of wind speed, a series of rotor power curves must be drawn (fig. 6.2). As a result the nett output curve is found, as well as the overall efficiency (from wind to water) of the system (fig. 6.2).
It can be seen that the resulting output curve is nearly a linear function of the wind speed. The overall efficiency varies strongly with the wind speed and in this example it reaches a clear maximum at \( V = 3 \) m/s. We shall define the wind speed at which the overall efficiency reaches a maximum as the design wind speed \( V_d \) of the system. In practice it is the wind speed at which \( C_p \) reaches its maximum value \( C_{p_{\text{max}}} \).

This design wind speed can also be calculated by realizing that at each wind speed, so also at \( V_d \), the nett power supplied by the rotor-pump combination must be equal to the hydraulic power to lift the water (5.8):

\[
\text{nett rotor-pump power} = \text{hydraulic power}
\]

\[
\eta_{\text{mech}} \frac{P_{\text{mech}}}{P_{\text{hyd}}} = \eta_{\text{mech}} \frac{C_p \frac{1}{2} \rho V^3 \pi R^2}{q \rho_w g H} = \eta_{\text{mech}} C_{p_{\text{max}}} \frac{1}{2} \rho V^3 \pi R^2 = q_d \rho_w g H \tag{6.1}
\]

at \( V = V_d \):

\[
\eta_{\text{mech}} C_{p_{\text{max}}} \frac{1}{2} \rho V^3 \pi R^2 = q_d \rho_w g H \tag{6.2}
\]

The flow \( q \) is equal to the ideal flow, determined by the stroke volume and the speed, multiplied by the volumetric efficiency of the pump:

\[
q = n_{\text{vol}} \frac{\pi s}{4} \frac{D^2}{\rho} \frac{\Omega}{2\pi} \tag{6.3}
\]

or

\[
q = n_{\text{vol}} \frac{\pi s}{4} \frac{D^2}{\rho} \frac{\lambda V}{2\pi R} \tag{6.4}
\]

Substituting (6.4) in (6.3) for \( V = V_d \) gives an expression for \( V_d \):

\[
V_d = \sqrt{\frac{n_{\text{vol}} \frac{s D^2}{\rho} \lambda \rho_w g H}{4 C_{p_{\text{max}}} \eta_{\text{mech}} \rho \pi R^3}} \tag{6.5}
\]
Fig. 6.2 Coupling of a rotor of the Tunesia pump of fig. 5.5.
Rotor data: diameter = 4 m, $C_p = 0.38$, $\lambda_d = 2$, $P_{max}$, $\lambda_{max} = 3.5$. 
With the data of fig. 6.2 (and fig. 5.5) we find:

\[
v_d = \sqrt{(0.98 \times 0.08 \times (0.141)^2 \times 2 \times 1000 \times 9.8 \times 11.4) / (4 \times 0.38 \times 0.85 \times 1.2 \times \pi \times 2^3)} = 2.99 \text{ m/s}
\]

Note: We can see from (6.5) that the design wind speed can easily be changed by changing the stroke of the pump, or by installing another size pump. A change in water lifting head also changes the design wind speed.

At high wind speeds, say above 10 m/s for example, the forces on the rotor and on the rest of the structure become quite high. Because the number of hours that these high wind speeds occur is small, their energy content is low. The windmill is protected against these wind speeds with a safety system that reduces the forces and the speed above a given speed, the so-called rated wind speed \( V_r \).

At very high wind speeds, say above 15 to 20 m/s, one prefers to stop the windmill completely to avoid damage. This wind speed is called the cut-out speed \( V_{out} \), or sometimes the furling speed (cf. furling the sails of a sail wind rotor).

These wind speeds are indicated in the power-wind speed curve of fig. 6.3.

![Fig. 6.3 The power-wind speed curve of a water pumping windmill.](image-url)
This power-wind speed curve is basically a wind tunnel-type of curve, i.e. it shows the power output in the hypothetical situation that the whole windmill is placed in a wind tunnel. This is because the original data, the $C_p-\lambda$ curve of the rotor and the $p-\omega$ curve of the pump, are derived from wind tunnel and laboratory experiments.

In practice, however, the windmill is subjected to a varying wind speed with a varying direction, resulting in lower outputs. Also the instantaneous wind speed and the instantaneous output are not well correlated because of the inertia of the system. So, during a short increase in wind speed, the output is still low, but when the wind is already slowing down, the windmill has gained momentum and produces more output, even when the wind speed at that same moment is low. This is why one usually takes the average values of both wind speed and output for periods of 10 minutes. A large number of output values in each wind speed interval are averaged to yield one value per interval (the so-called "bin method").

The result is an actual power-wind speed curve more or less similar in shape but usually lower than the wind tunnel curve (fig. 6.3).

6.2 Mathematical description windmill output

In order to simplify the mathematical description of the output of a water pumping windmill we shall use two assumptions:

1. The average torque of the pump is constant. In fig. 5.5 we have seen that this is a reasonable assumption.

2. The $C_Q-\lambda$ characteristic of the wind rotor is linear. As shown in fig. 4.4 this is true from tip speed ratios slightly below $\lambda_d$ up to $\lambda_{\text{max}}$. So only the starting behaviour cannot be described when we use this assumption.
The first assumption implies that the torque produced by the rotor at speed $V$ must be equal to the (design) torque produced at $V_d$:

$$C_Q \frac{1}{2} p A V^2 R = C_{Q_d} \frac{1}{2} p A V_d^2 R$$  \hspace{1cm} (6.6)

or:

$$\frac{C_Q}{C_{Q_d}} = \frac{V_d^2}{V^2}$$  \hspace{1cm} (6.7)

The second assumption can be written algebraically:

$$C_Q = \frac{C_{Q_d}}{\lambda_{\text{max}} - \lambda_d} \ast (\lambda_{\text{max}} - \lambda)$$  \hspace{1cm} (6.8)

Substituting relation (6.7) in (6.8) gives:

$$\frac{V_d^2}{V^2} = \frac{\lambda_{\text{max}} - \lambda}{\lambda_{\text{max}} - \lambda_d}$$  \hspace{1cm} (6.9)

which can be rewritten into:

$$\frac{\lambda}{\lambda_d} = \frac{\lambda_{\text{max}}}{\lambda_d} - \frac{V_d^2}{V^2} \left( \frac{\lambda_{\text{max}}}{\lambda_d} - 1 \right)$$  \hspace{1cm} (6.10)

The output power $P(V)$ now can be found with:

$$\frac{P(V)}{P_d} = \frac{Q_d}{Q_d} \frac{\Omega}{\Omega_d} = \frac{\Omega}{\Omega_d} = \frac{V}{V_d} * \frac{\lambda}{\lambda_d}$$  \hspace{1cm} (6.11)
with (6.10) the power output becomes:

\[
\frac{P(V)}{P_d} = V \frac{\lambda_{\text{max}}}{\lambda_d} \left( \frac{\lambda_{\text{max}}}{\lambda_d} - 1 \right)
\]  

(6.12)

This function is shown in fig. 6.4 for three values of \(\lambda_{\text{max}}/\lambda_d\).

Note that the starting wind speed found from (6.12) by taking \(P(V) = 0\) is meaningless, because the \(C_Q-\lambda\) characteristic is not linear anymore for low \(\lambda\)-values.

The typical shape of the \(C_p-n-V\) curve in fig. 6.2 can now be found mathematically by realizing that for a constant \(n\):

\[
\frac{P(V)}{P_d} = \frac{C_p V^3}{C_{p_{\text{max}}} V_d^3}
\]  

(6.13)

with (6.12) this gives us:

\[
\frac{C_p}{C_{p_{\text{max}}}} = \frac{V_d^2}{V^2} \frac{\lambda_{\text{max}}}{\lambda_d} \left[ 1 - \frac{V_d^2}{V^2} \left( 1 - \frac{\lambda_d}{\lambda_{\text{max}}} \right) \right]
\]  

(6.14)

This function is shown in fig. 6.5.
The output of a water pumping windmill, related to its output at the design wind speed $V_d$, as a function of the ratio $V/V_d$ for different values of $\lambda_{\text{max}}/\lambda_d$. The torque of the pump is assumed to be constant and the rotor $C_Q-\lambda$ curve is assumed to be linear.
Fig. 6.5 The power coefficient of the rotor of a water pumping windmill coupled to a constant torque pump with a constant efficiency, related to the $C_{p_{\text{max}}}$ of the rotor, as a function of $V/V_d$ for different values of $\lambda_{\text{max}}/\lambda_d$. The rotor of the windmill is assumed to have a linear $C_Q^\lambda$ characteristic.
6.3 Starting behaviour

The simplest description of the starting behaviour of a water pumping windmill is the static description, in which the starting torque of the rotor is equal to the maximum torque required by the pump, at the starting wind speed $V_{st}$:

$$Q_{rotor} = Q_{pump}^{(max)}$$ (6.15)

$$C_{Q_{st}} \frac{1}{2} \rho \frac{V_{st}^2}{A R} = \frac{1}{2} \rho \omega g H \frac{\pi}{4} \frac{D^2}{p}$$ (6.16)

Remember that the maximum torque of the pump is $\pi$ times its average torque (see formula 5.4) and that the average torque is equal to the torque $Q_d$ produced by the rotor at its design wind speed (see formula 6.6). This leads to:

$$C_{Q_{st}} \frac{1}{2} \rho \frac{V_{st}^2}{A R} = \pi \frac{C_p^{max}}{\lambda_d} \frac{1}{2} \rho \frac{V_d^2}{A R}$$ (6.17)

or:

$$V_{st} = V_d \sqrt{\frac{\pi C_p^{max}}{\lambda_d C_{Q_{st}}}}$$ (6.18)

In the case of a multi-bladed rotor with $\lambda_d = 1$, $C_p^{max} = 0.35$ and $C_{Q_{st}} = 0.5$ this gives us: $V_{st} = 1.48 \times V_d$. In other words, the windmill needs a gust of wind, with a velocity about 1.5 times the design wind speed, to be able to start. Later on we shall see that the effect of the rotor inertia will reduce this factor somewhat, but still the starting wind speed will usually be higher than the design windspeed.

* The first analysis of the starting behaviour of windmills is given by J. van Meel in his "Notes on piston pumps coupled to wind rotors: inertia and leaks", (in Dutch), internal report R 294 D Eindhoven University of Technology, The Netherlands, 1977.
The static description fails to explain why in reality windmills will start at lower wind speeds and also why they show an oscillating behaviour at wind speeds below the actual starting wind speed. The main factors to be included are the inertia of the rotor and the unbalance due to the finite weight of the piston, pump rod and crank.

Minor effects are the friction of the bearings, the friction of the piston in the cylinder, the water losses due to leaking between piston and cylinder and the inertia forces due to the acceleration and deceleration of the water column and the mass of piston pump rod and crank. We assume that the friction of the bearings is included by slightly reducing the rotor torque. The friction of the piston in the cylinder is usually neglected, but can be added to (upward stroke) or subtracted from (downward stroke) the weight causing the unbalance. The inertia forces are normally very small, because the rotor speeds are still very low. Only with excessively long suction or delivery lines without airchambers they can become important.

We conclude that the following torque equation describes the behaviour of the system:

\[ \tau_{\text{rotor}} = \tau_{\text{pump}} + \tau_{\text{rotor inertia}} + \tau_{\text{unbalance}} \quad (6.19) \]

Introducing the position angle \( \theta = \Omega t \), the rotor inertia \( I \) and the unbalanced weight \( G \) we can write (6.19) as:

upward stroke: \( \tau_r = Q_d \pi \sin \theta + I \frac{d\Omega}{dt} + \frac{1}{2} s G \sin \theta \) \quad (6.20) \\
downward stroke: \( \tau_r = 0 + I \frac{d\Omega}{dt} + \frac{1}{2} s G \sin \theta \)

We change to dimensionless quantities by dividing by \( Q_d \) and for a moment we shall concentrate on the equation for the upward stroke only:

\[ \frac{\tau_r}{Q_d} = \pi \sin \theta + \frac{I}{Q_d} \frac{d\Omega}{dt} + \frac{1}{2} \frac{s G}{Q_d} \sin \theta \quad (6.21) \]
Introducing the time constant \( \tau \) with:

\[
\tau = \sqrt{\frac{1}{Q_d}}
\]  

(6.22)

we can change (6.21) into:

\[
\frac{Q_r}{Q_d} = (\pi + \frac{ksG}{Q_d}) \sin \theta + \tau^2 \frac{d\Theta}{dt}
\]  

(6.23)

Rewriting this slightly and adding the equation for the downward stroke gives the complete dimensionless description:

upward stroke:

\[
\frac{d\Theta}{dt} = \frac{Q_r}{Q_d} - (\pi + \frac{ksG}{Q_d}) \sin \theta
\]  

(6.24)

downward stroke:

\[
\frac{d\Theta}{dt} = \frac{Q_r}{Q_d} - \frac{ksG}{Q_d} \sin \theta
\]

This set of differential equations can be solved numerically. A simple numerical procedure that can be handled by calculators, is given below. If the differential equations are solved for \( G = 0 \), the result is that for \( Q_r/Q_d = 0.7 \pi \) the rotor just manages to pass the most difficult position, i.e. \( \theta = 3\pi/4 \), during the first revolution. The next revolutions are easier, because the rotor arrives at the lowest position \( \theta = 0 \) with a given speed. This is shown in the example of fig. 6.6. Note that in this model the inertia does not affect the reduction in torque required to start the rotor. It only affects the acceleration rate of the rotor.

With increasing values of \( G \) the ratio \( Q_r/Q_d \) increases from \( 0.7 \pi \) to higher values. This is shown in fig. 6.6 with \( \frac{ksG}{Q_d} = 0.45 \), leading to \( Q_r/Q_d = 2.61 = 0.83 \pi \). This is why windmill designers will counterbalance the weight of piston rod and piston, in order to reduce \( \frac{ksG}{Q_d} \) to a low value.
The numerical procedure to solve (6.24) is as follows. The dimensionless time \( t/\tau \), called \( t' \) here, is increased with small steps, equal to \( \Delta t' \) and alternatively new values of \( \Omega t \) and \( \theta \) are calculated.

**starting values**

\[
\begin{align*}
t' &= 0 \\
\Omega t &= (\Omega t)_{\text{begin}} \\
t' &= \frac{1}{2} \Delta t \\
\theta &= \theta_{\text{begin}} \\
t' &= \Delta t'
\end{align*}
\]

**Loop:**

\[
\text{start}
\]

- **positive** \( \Delta \Omega t = \Delta t' \left( \frac{q_r}{Q_d} - \left( \pi + \frac{k sc}{Q_d} \right) \sin \theta \right) \)

- \( \sin \theta = ? \)

- **negative** \( \Delta \Omega t = \Delta t' \left( \frac{q_r}{Q_d} - \frac{k sc}{Q_d} \sin \theta \right) \)

\[
\begin{align*}
\Omega t &= \Omega t + \Delta \Omega t \quad \text{(print \( \Omega t \))}

t' &= t' + \frac{1}{2} \Delta t' \\
\Delta \theta &= \Delta t' \times \Omega t \\
\theta &= \theta + \Delta \theta \quad \text{(print \( t \))}

t' &= t' + \frac{1}{2} \Delta t'
\end{align*}
\]
Fig. 6.6 The starting behaviour of a given water pumping windmill with a constant rotor torque $Q_r$, just high enough to prevent a stop of the rotor at $\theta = \frac{3\pi}{4}$. 

\[
\frac{Q_r}{Q_d} = 2.61 \\
\frac{1}{2}sG = 0.45 \\
\frac{1}{4} \Delta t/r = 0.01
\]
6.4 Piston pumps with a leakhole

In order to improve the starting characteristics of a windmill equipped with a reciprocating piston pump one can drill a very small hole in the piston (or the valve). The effect of this leakhole is that at very low speeds, i.e. at starting, all water that could be pumped is leaking through the hole. This implies that the pressure on the piston is very low and as a result the starting torque required is low. If the speed is high, then the quantity of water leaking through the hole is small compared to the normal output of the pump, and the pump behaves as a normal piston pump.

In this section we will describe the characteristics of such pumps and the effect they have on the starting behaviour of a windmill.

In our analysis we shall use the quantities as indicated in fig. 6.7. For most leakholes the length $l$ is only a few times the diameter $d$, or sometimes even smaller than $d$. This means that pipe flow formulas cannot be used, but that the expressions for orifice flow must be used. The pressure difference over the leakhole is:

$$P_1 - P_2 = f \frac{k}{2} \rho_w C^2$$

(6.25)

Fig. 6.7 Schematic drawing of a piston with a leakhole
The friction factor $f$ is slightly dependent on the Reynolds number but for values of $\text{Re} > 10^4$ a value of $f = 2.75$ is a good approximation. We shall use the following reference values for torque and flow of the ideal pump:

- Average torque: $\overline{Q}_{id} = \frac{1}{\pi} \rho_w g H \frac{1}{2} V_s$
- Instantaneous torque: $Q_{id} = \overline{Q}_{id} \sin \theta$
- Average flow: $\overline{q}_{id} = \frac{Q}{2\pi} V_s$
- Instantaneous flow: $q_{id} = \overline{q}_{id} \sin \theta$
- Stroke volume: $V_s = \frac{\pi}{4} D_p^2$

At low piston speeds the velocity $C$ of the flow in the leakhole is given by the continuity of mass flow:

$$C = \frac{D_p^2}{d^2} V_p$$  \hspace{1cm} (6.26)

If the speed of the piston increases, $C$ will increase and consequently the pressure $p_1 - p_2$ to sustain the flow. At a given speed $V_p = V_o$ the pressure difference $p_1 - p_2$ equals the pressure head:

$$p_1 - p_2 = \rho_w g H$$  \hspace{1cm} (6.27)

With (6.27) the speed in the leakhole becomes:

$$C = \sqrt{\frac{2 g H}{f}}$$  \hspace{1cm} (6.28)

In other words, the discharge of the pump starts if the speed of the piston $V_p > V_o$ with:

$$V_o = \frac{d^2}{D_p^2} \sqrt{\frac{2 g H}{f}}$$  \hspace{1cm} (6.29)
We know from fig. 5.6 that the speed is at its highest value if \( \theta = \frac{\pi}{2} \), so the minimum rotational speed to pump water is given by:

\[
\Omega_o = \frac{V_0}{\frac{1}{2} s} = \frac{d^2}{\frac{1}{2} s D^2_p} \sqrt{\frac{2gH}{f}}
\]  

(6.30)

At higher (constant) rotational speeds the discharge starts at angles \( \theta_o \) smaller than \( \frac{\pi}{2} \) (see fig. 6.8)

\[
\Omega \frac{1}{2} s \sin \theta_o = \Omega_o \frac{1}{2} s
\]

\[
\theta_o = \arcsin \frac{\Omega_o}{\Omega}
\]

(6.31)

Fig. 6.8 The piston speed \( V_p \) exceeds the critical speed \( V_o \) for discharge at position angles between \( \theta_o \) and \( \pi - \theta_o \).
The torques to drive the pump at speeds below or above \( \Omega_0 \) are calculated as follows:

\[
Q_p = (p_1 - p_2) \times \frac{1}{2} v_s \times \sin \Theta (\Omega < \Omega_0) \tag{6.32}
\]

With (6.25), (6.26) one finds:

\[
Q_p = \frac{1}{2} \rho \frac{p}{d^4} v^2_p \times f \times \frac{1}{2} v_s \times \sin \Theta
\]

With \( v_p = \Omega \frac{1}{2} s \sin \Theta \), (6.29) and (6.30) this transforms into:

\[
Q_p = \rho \frac{g H \times \Omega^2}{\Omega_0^2} \times \frac{1}{2} v_s \times \sin^3 \Theta
\]

or

\[
Q_p = \bar{Q}_{id} \times \pi \times \frac{\Omega^2}{\Omega_0^2} \times \sin^3 \Theta (\Omega < \Omega_0) \tag{6.33}
\]

The average torque \( \bar{Q}_p \) for \( \Omega < \Omega_0 \) can be calculated with:

\[
\frac{1}{2\pi} \int_0^\pi \sin^3 \Theta \, d\Theta = \frac{2}{3\pi}
\]

resulting in:

\[
\bar{Q}_p = \bar{Q}_{id} \times \frac{2}{3} \frac{\Omega^2}{\Omega_0^2} (\Omega < \Omega_0) \tag{6.35}
\]

For rotational speeds above \( \Omega_0 \), we have to divide the stroke in the parts without discharge, i.e. between 0 and \( \Theta_0 \), and between \( \pi - \Theta_0 \) and \( \pi \), and the part with discharge, i.e. between \( \Theta_0 \), and \( \pi - \Theta_0 \). Without discharge the torque is given by the expression (6.33) found above. With discharge the torque is simply the instantaneous torque \( Q = \bar{Q}_{id} \pi \sin \Theta \). The average torque becomes:
\[ \bar{Q}_p = \frac{\bar{Q}_{id}}{2\pi} \left[ \int_{0}^{\pi/2} \frac{\Omega^2}{\Omega_0^2} \sin^3 \theta \, d\theta + \int_{0}^{\pi} \frac{\Omega^2}{\Omega_0^2} \sin \theta \, d\theta \right] (6.36) \]

The result is:

\[ \bar{Q}_p = \bar{Q}_{id} \left\{ \frac{2}{3} \frac{\Omega^2}{\Omega_0^2} + \frac{2}{3} (1 - \frac{\Omega^2}{\Omega_0^2}) \times \sqrt{1 - \frac{\Omega^2}{\Omega_0^2}} \right\} \quad (\Omega > \Omega_0) \quad (6.37) \]

The instantaneous torques are shown in fig. 6.9 below.

---

**Fig. 6.9** Torque fluctuations in a piston pump with a leakhole, for rotational speeds below and above the critical discharge speed \( \Omega_0 \).
The discharge of the pump, taking place between $\Theta_o$ and $\pi - \Theta_o$ only, has to be subtracted by the flow through the leakhole:

$$q_{\text{leak}} = \frac{\pi}{4} d^2 * C$$  \hspace{1cm} (6.38)

With (6.28) this becomes:

$$q_{\text{leak}} = \frac{\pi}{4} d^2 * \sqrt{\frac{2 g H}{f}}$$  \hspace{1cm} (6.39)

With (6.29) and (6.30) this reduces to:

$$q_{\text{leak}} = \frac{\Omega_o}{2} \frac{1}{\sqrt{g}} \sqrt{s}$$  \hspace{1cm} (6.40)

or with

$$q_{\text{id}} = \frac{\Omega_o}{2\pi} \sqrt{s}$$

$$q_{\text{leak}} = q_{\text{id}} * \frac{\Omega}{\Omega}$$  \hspace{1cm} (6.41)

The instantaneous flow discharged by the pump becomes (fig. 6.10)

$$q = q_{\text{id}} (\pi \sin \Theta - \pi \frac{\Omega}{\Omega})$$  \hspace{1cm} (6.42)

The average flow is found after integration:

$$\bar{q} = \frac{q_{\text{id}}}{2\pi} \int_{\Theta_o}^{\pi-\Theta_o} (\pi \sin \Theta - \pi \frac{\Omega}{\Omega}) \, d\Theta$$
resulting in:

\[
\bar{q} = \bar{q}_{id} \left[ \sqrt{1 - \frac{\Omega^2}{\Omega_0^2}} - \frac{\Omega}{\Omega_0} \left( \frac{\pi}{2} - \Theta_0 \right) \right] \tag{6.43}
\]

The volumetric efficiency due to the effect of the leakhole is equal to the ratio of \( \bar{q} \) and \( \bar{q}_{id} \):

\[
\eta_{\text{leak, vol}} = \sqrt{1 - \frac{\Omega^2}{\Omega_0^2}} - \frac{\Omega}{\Omega_0} \left( \frac{\pi}{2} - \arcsin \frac{\Omega}{\Omega_0} \right) \tag{6.44}
\]

The mechanical efficiency due to the effect of the leakhole is found with equation (5.8):

\[
\eta_{\text{leak, mech}} = \eta_{\text{leak, vol}} \frac{\bar{Q}_{id}}{\bar{Q}_p} \tag{6.45}
\]

The expressions for the average torque (6.37) and for the efficiencies are shown in fig. 6.11.
Fig. 6.11 The average torque and efficiencies of a piston pump, due to the effect of a leakhole, as a function of the relative speed. The rotational speed $\omega_0$ is the speed at which the pumps starts pumping water.
6.5 Starting behaviour including leakhole

When describing the starting behaviour of a windmill equipped with a pump with leakhole, we must realise that the rotational speed is not constant anymore. The condition for discharge remains:

\[ V_p > V_0 \]  

(6.46)

This leads to:

\[ \frac{1}{2} s \sin \theta > \frac{\Omega_o}{\Omega} \frac{1}{2} s \]

or:

\[ \Omega > \frac{\Omega_o}{\sin \theta} \]  

(6.47)

The torque required for speeds below or above this critical speed are given by equation (6.33) and (5.2). They change the torque equations for rotor plus pump in the following manner (see also 6.20):

**upward stroke**

\[ \Omega < \frac{\Omega_o}{\sin \theta} : Q_r = Q_d + \frac{\Omega^2}{\Omega_o^2} \sin^3 \theta + I \frac{d\Omega}{dt} + \frac{1}{2} s G \sin \theta \]

\[ \Omega > \frac{\Omega_o}{\sin \theta} : Q_r = Q_d + \sin \theta + I \frac{d\Omega}{dt} + \frac{1}{2} s G \sin \theta \]  

(6.48)

**downward stroke**

\[ Q_r = 0 + I \frac{d\Omega}{dt} + \frac{1}{2} s G \sin \theta \]
Rewriting these equations into dimensionless quantities (compare with (6.24)) gives us:

**upward stroke**

\[
\begin{align*}
\Omega < \frac{\Omega_0}{\sin \theta} & : \quad \frac{d\Omega}{Q_d} = \frac{Q_r}{Q_d} - \pi \frac{\Omega_0^2}{\Omega_0^2} \sin^3 \theta - \frac{k_s G}{Q_d} \sin \theta \\
\Omega > \frac{\Omega_0}{\sin \theta} & : \quad \frac{d\Omega}{Q_d} = \frac{Q_r}{Q_d} - \frac{k_s G}{Q_d} \sin \theta - \frac{\sin \theta}{Q_d}
\end{align*}
\] (6.49)

**downward stroke**

\[
\frac{d\Omega}{Q_d} = \frac{Q_r}{Q_d} - \frac{k_s G}{Q_d} \sin \theta
\]

In order to determine the behavior of the system at varying wind speeds the term \(Q_r/Q_d\) must be worked out further.

\[
\frac{Q_r}{Q_d} = \frac{C_Q(\lambda) \eta \frac{1}{2} \rho v^2 \pi A R}{C_{Q_d} \eta \frac{1}{2} \rho v^2 \pi A R} = \frac{C_Q(\lambda)}{C_{Q_d}} \frac{v^2}{v_d^2}
\] (6.50)

in which \(C_Q(\lambda)\) is given as an analytical function (or a table) and the tip speed ratio is calculated with \(\lambda = \frac{VR}{v}\).

The differential equations must be solved numerically.
6.6 Calculating the diametre of a leakhole

In order to relate the leakhole diametre with the design speed of the windmill we assume that at the design speed the leak flow is only 10% of the design flow, in other words the volumetric efficiency is 90%:

$$\eta_{\text{leak, vol}} = 0.90$$ \hfill (6.51)

In fig. 6.11 it can be seen (or calculated from equation (6.44)) that this value will be reached when (Note: \( \Omega = \Omega_d \)):

$$\frac{\Omega_d}{\Omega_0} = 15.4 \ (15.38281 \ \text{to be exact})$$ \hfill (6.52)

The design speed \( \Omega_d \) is found with expression (6.5) for the design wind speed \( V_d \):

$$\Omega_d = \frac{\lambda_d}{R} \sqrt{\frac{\eta_{\text{vol}} s D^2 \lambda_d \rho w g H}{4 C_{p_{\text{max}}} \eta_{\text{mech}} \rho w R^3}}$$ \hfill (6.53)

With equation (6.30) for the speed at which pumping starts (6.52) can be rewritten into:

$$\frac{2 d^2}{s D^2 P} \sqrt{\frac{2 g H}{P}} = \frac{\lambda_d}{15.4 R} \sqrt{\frac{\eta_{\text{vol}} s D^2 \lambda_d \rho w g H}{4 C_{p_{\text{max}}} \eta_{\text{mech}} \rho w R^3}}$$ \hfill (6.54)

Rearranging yields an expression for the leakhole diametre:

$$d^2 = \frac{P^3}{30.8} \sqrt{\frac{s^3 \lambda^3}{C_{p_{\text{max}}} \eta_{\text{mech}} R^5}} \frac{\rho w}{8 \pi \rho}$$ \hfill (6.55)
This expression can be simplified with the following values:

\[ \rho_w = 1000 \, \text{kg/m}^3 \]

\[ f = 2.75 \, \text{(submerged orifice flow)} \]

\[ \rho = 1.2 \, \text{kg/m}^3 \]

This gives:

\[ d^2 = 0.31 \frac{D^3}{\rho} \sqrt{\frac{\eta_{vol} s^3 \lambda_d^3}{C_p \eta_{mech} R^5 \max}} \]  \hspace{1cm} (6.56)

Example: WEU-I windmill

\[ R = 1.5 \, \text{m} \]

\[ C_p \eta_{mech} = 0.36 \times 0.6 = 0.216 \]

\[ \eta_{vol} = 0.95 \times 0.9 = 0.855 \]

\[ \lambda_d = 2 \]

The design formula (6.56) becomes:

\[ d^2 = 0.633 \frac{D^3}{\rho} \sqrt{s^3} \]  \hspace{1cm} (6.57)

With a piston diameter of 0.1 m and a stroke of 0.1 m the leakhole must have a diameter of 4.5 mm.
7. GENERATORS

An extensive description of the working principles of a generator is given in the literature \[24, 25\].

For this text a short survey of different types of electric machines* is sufficient.

There are three main types:

1) The synchronous machine : Widely used as generator; as a motor for accurate constant speed applications.

2) The asynchronous machine : The induction squirrel-cage motor is of this class.

3) The commutator machine : For example a DC-motor or an (old type) car dynamo.

We shall give a short description of these three types and shall discuss their characteristics with special reference to their aptitude of being driven by a wind rotor.

7.1. The synchronous machine (SM)

This type is usually constructed in the following way:

- The rotor consists of a number of poles, around which coils are wound. When a DC current (the field current or excitation current) is flowing through the coils, magnetic poles are created. The number of poles is even (each pair consists of a South and a North pole and will usually have a value between 2 and 24. When the number of pairs of poles is \( p \) and the rotor rotates with \( n_C \) r.p.m., then a fixed point on the stator will see a magnetic field periodically changing with a frequency of \( p n_C \).

* The first version of this chapter is taken from SWD publication 78-3 "Matching of wind rotors to low power electrical generators" by H.J. Hengeveld, E.H. Lysen and L.M.M. Paulissen, 1978.
On the stator, normally three coils are wound in such a way that, when a three phase current system flows through these coils (with a certain frequency $f$), a rotating magnetic field is generated. If the rotor and stator-field rotate at the same frequency, but only then, a non-pulsating torque is exerted by the one field on the other. In that case applies $f = p n_G$.

When the stator of the SM is connected to a voltage system with a fixed frequency, $f$, the shaft (after synchronisation) will rotate at a fixed speed of 60 $f/p$ revolutions per minute. Vice versa applies that, when the rotor rotates at a fixed speed, the SM supplies a voltage of a fixed frequency. As a result, a wind rotor coupled to a synchronous machine has to rotate at a constant speed (the synchronous speed) if the machine is directly connected to the public grid. If the machine operates independently, then speed variation is possible, but the output voltage will have a variable frequency. For electric heating this will present no difficulties, for other applications rectification and subsequent DC/AC conversion might be necessary.

In general, the rotor of the SM has two slip rings to which the field current (DC) can be fed. The generated voltage and current is taken from a number of stator coils (depending on the number of phases). In fig. 7.1 this is drawn schematically.

![Fig. 7.1 Schematic representation of a three-phase synchronous machine.](image-url)
- There also exist slip ringless (or brushless) types of synchronous generators.
In one type a small extra generator is mounted on the extended shaft of the synchronous generator. This generator normally has field coils in the stator and the current is generated in the rotor.
The generated current is rectified by diodes (mounted on the shaft) and fed to the field coils on the rotor of the original synchronous generator. Older types possess a small DC generator as an extra generator.
Another brushless type of synchronous generator is the generator with a permanent magnet rotor.
Advantages: - No losses caused by excitation currents.
- No brushes, therefore lower friction losses.
Disadvantages: - A permanent field is not as strong as an excited field.
- The possibility of controlling the generator output by controlling the field current is eliminated.
- Higher starting torque.

- NOTE:
Synchronous machines can also have their field poles on the stator, such that the main current is generated in the rotor.

7.2. The asynchronous machine (AM)

- Basically, the stator of the AM is the same as that of the SM. The stator coils are normally connected to an AC-voltage system, e.g. grid. These coils, one for a single-phase AM and three for the three-phase AM, will supply the rotating magnetic field.

- The rotor windings are generally not connected to a power source but are short-circuited. Either a squirrel-cage rotor is used or the rotor windings are short-circuited outside the machine. The terminals are led outwards by means of slip rings. The latter construction gives the possibility to control the machine.
The rotating stator field induces currents in the rotor. These currents are only limited by the impedance of the rotor winding. The magnetic field in the stator exerts a torque on the current conducting windings of the rotor and the rotor will have to rotate, forced by this torque. When the rotor rotates at the same speed as the rotating stator field (this speed is called the synchronous speed), no current is induced in the rotor and no torque is exerted on the rotor by the stator field. This means that, if the stator has to exert a force on the rotor, the mechanical rotor speed \( \omega_m \) has to differ from the speed of the stator field \( \omega_s \): the rotor rotates at an asynchronous speed with respect to the speed of the stator field.

This speed difference is expressed in the relative "slip" \( s \) of the machine:

\[
s = \frac{\omega_s - \omega_m}{\omega_s}
\]

A practical value of \( s \) is 4%.

When an AM, rotating at synchronous speed, is connected to a load requiring a torque, the rotor speed will decelerate to a value where the difference in the speed of rotor field and stator field causes enough rotor current to produce the required torque. Now the machine acts as a motor.

When, on the contrary, the AM is driven by a prime mover at a speed higher than the synchronous speed, also currents will be generated in the rotor (the stator is connected to our existing fixed frequency supply.) These currents excite a magnetic field which generates a voltage and subsequently a current in the stator windings. Then the machine acts as a generator: electric power is leaving the stator connections. The function of the stator windings is:
1) to produce a rotating magnetic field,
2) to conduct the generated power.
If no three-phase voltage system is available, the machine will not easily operate as a generator, because it cannot generate its own field current in the rotor.

In fig. 7.2 a possible configuration is drawn of an AM working as a generator without a connection to a public grid.

![Schematic representation of an asynchronous machine (AM), equipped with capacitors to provide self-excitation.](image)

The stator windings form oscillating circuits together with the extra capacitors. These circuits are tuned to the desired frequency (50 Hz for instance). When the speed corresponding to this frequency is reached, the remanent magnetism of the rotor is sufficient to induce a voltage in the stator coils. Because of the L-C circuits reactive currents can flow in the stator coils which on their turn induce currents in the rotor. These currents will produce the required rotating magnetic field.

7.3. **Comparison of the SM and AM**

A rough comparison of the SM and AM can be made by their torque-speed curves as they occur in connection with a strong grid with fixed frequency (see fig. 7.3 and 7.4).
Fig. 7.3
The torque speed curve of a synchronous machine coupled to a strong grid.

- The synchronous machine can operate only at synchronous speed (fig. 7.3). At this speed all torque values between $+Q_{\text{max}}$ and $-Q_{\text{max}}$ can be demanded from or applied to the shaft. If the torque exceeds $Q_{\text{max}}$, the machine will no longer keep pace with the network frequency. Large pulsating torques and currents are brought about in that case, and these may possibly damage the machine.

A fixed-frequency network thus seriously limits the generator speed to one value. As a result, starting of the machine requires a special procedure.Disconnected from the grid, the machine has to be speeded up to synchronous speed by means of an auxiliary motor. When the right polarity, voltage phase sequence and the frequency are checked with special equipment, the connection with the network can be made. If no strong grid is available and the SM has to operate as a generator, then the rotational speed must be controlled mechanically (for instance: the speed of the diesel engine, the steam supply, the transmission ratio) or an AC/DC/AC converter must be applied.
The asynchronous machine can operate at a certain range of speed values around the synchronous speed $n_0$. As we have seen, the origin of the transfer of energy between electrical and mechanical power is a certain difference (slip) between the rotational and the synchronous speed. At synchronous speed (slip is zero) no torque is exchanged between the machine and the load. On the other hand, when the slip is too large, the maximum or minimum value of the torque is exceeded and the machine will decelerate to zero in the motor mode. In the generator mode the machine will run free and speed up, limited only by the mechanical friction.

A fixed-frequency network thus makes possible a small range of stable values of $n$. The AM can be started as a motor by simply connecting the stator to the network. Sometimes special arrangements have to be made to prevent high currents during starting. The range with stable $n$-values can be enlarged by several methods. One method is to use slip rings on the rotor by means of which the rotor windings can be short-circuited through a variable resistance. The higher this resistance, the flatter the torque speed curve. An example is given in figure 7.4 by the dotted line. From this it is clear that the band with stable $n$-values is enlarged.

If no strong network is available, the AM can be used with the arrangements presented in figure 7.2. In this case the rotational speed should be kept within the range indicated in fig. 7.4 to obtain a fixed output frequency.

At low wind speeds, when the wind rotor produces little torque, both machines tend to swing between generator and motor mode. Precautions have to be taken to prevent the occurrence of the motor mode as much as possible.

The efficiency of synchronous machines is usually better (approximately 10%) than that of asynchronous machines.
7.4. The commutator machine (CM)

Generally the commutator machine is constructed as follows:

- The stator is equipped with one or more pairs of poles to generate the magnetic field. The field can be obtained by electric magnets or by permanent magnets.

- On the rotor a number of coils is distributed in slots. The coils are connected to the segments of a commutator. Brushes resting on the commutator conduct the current to the outside world. The voltage generated is a DC voltage with a small ripple caused by the commutation.

The CM is one of the oldest types of electrical machines and has been used extensively. The extra maintenance of the commutator, however, has favoured the AM and SM more and more. For generating purposes, the CM has been superseded by the synchronous machine, although with the advent of the DC/AC converter there still remains a role for the CM. For constant speed driving applications the AM has taken over, but for variable speed drives the CM is still used, e.g. most electric trains have CM drive. The torque-speed curves strongly depend on voltage and field current, as shown in fig. 7.5

![Torque-speed curve](image)

**Fig. 7.5** The torque-speed curve of a (separately excited) commutator machine at two voltages (a) and at two values of the field current (b).
7.5. Applications for wind energy

Nearly every type of generator has been utilized for wind energy applications. There is no "best type" of generator to be used up till now and we will limit ourselves to a brief indication of current developments.

For direct connection to the public grid the AM is quite favourite, because of the relatively simple synchronization procedure and the low cost and low maintenance requirements. Most small to medium scale wind turbines (10 - 100 kW) in Denmark and the Netherlands are equipped with an AM. The more or less constant speed, however, causes large torque variations and consequently large current variations. The first are not appreciated by the mechanical construction, the second not by the electric utility. Synchronous generators are used as well, but they show even more torque and current fluctuations. Damping these fluctuations is possible via mechanical means, such as flexible couplings, or by allowing the rotor to run at variable speeds. This can be accomplished with variable speed gearboxes or electrically with the use of AC/DC/AC convertors.

The latter method receives strong support in the Netherlands and is being tested on several machines, both with synchronous machines and DC commutator machines. Variable speed operation, but nevertheless a constant frequency and voltage output, can also be accomplished with special electrical machines.

For example in the double-fed AM the rotor is fed via slip rings with a current of a frequency \( \Delta f \), which is the difference between stator field speed and rotor speed.

For battery charging the DC commutator machine is still used extensively on the small wind turbines (wind charger, Aerowatt) but the synchronous machine with rectifiers is used more and more (similar to the development in automobiles).
If the power-speed characteristics of both the wind rotor and the generator are known, then the coupling procedure is rather straightforward and nearly identical to the coupling of a pump and a wind rotor. The only exception being that, because of the high speed required for most generators, a gearbox is usually necessary, of which the optimum gearing ratio has to be determined.

The problem is more complex than it seems however, because the power-speed relation depends upon the type of generator, the power factor* and the magnitude of the load, the field current, and upon the fact whether the speed of the machine is kept constant by a grid or is allowed to vary. An extra complication is that most generators are designed to operate at one optimum speed only, and it is often difficult to find their characteristics at lower speeds.

We shall leave these complications out of consideration for a moment and assume that we possess the power-speed curve of a generator, as well as that of a rotor.

8.1. Generator and wind rotor with known characteristics

When both the generator and wind rotor curves are known, the only variable left is the transmission ratio of the gearbox. This gearbox is necessary to increase the rotor speed to a value suitable to drive the generator (1000 or 1500 r.p.m. usually). So we can draw a number of power-speed curves of the generator for different transmission ratios i, in order to find one which is closest to the optimum power curve of the wind rotor (fig. 8.1) for wind speeds around the average wind speed of the location.

*) The power factor cos$\Phi$ is defined as the ratio of actual power and apparent power (watts/volt-amperes).
Fig. 8.1 A wind rotor coupled to three different generators:

(A) Fixed speed synchronous (----) and asynchronous (-- --) generator directly coupled to the grid

(B) Variable speed synchronous generator plus AC/DC/AC converter

(C) Variable speed commutator machine.
Once the optimum transmission ratio is found, the electrical output curve is drawn and the power output–wind speed relation of the system is found (fig. 8.2).

Fig. 8.2 Finding the power output–wind speed relation of a generator coupled to a wind rotor.

The P(V) curve of fig. 8.2 is a windtunnel-type of curve, because the rotor data are derived from windtunnel tests. The actual P(V) curves as measured in the field will be somewhat lower than the windtunnel curves, because of the variations in wind speed and wind direction. An example is given in fig. 8.3 showing also the rather scattered nature of the actual measurements.
8.2. **Designing a rotor for a known variable speed generator**

In many cases both the generator and the wind rotor characteristics are not available, yet one wishes to design a wind rotor for a given generator. In this case additional information is needed, i.e. the cut-in speed $V_{in}$ and the rated speed $V_r$ required.

They can be derived from energy output considerations, as we will see in chapter 9.

We conclude that the following known and unknown parameters are involved:

<table>
<thead>
<tr>
<th>known parameter</th>
<th>unknown parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>generator</td>
<td>$P_r$, $n_r$, $n_G(n_r)$</td>
</tr>
<tr>
<td></td>
<td>$P_{mech}(n_{in})$, $n_{in}$</td>
</tr>
<tr>
<td></td>
<td>$Q_{start}$</td>
</tr>
<tr>
<td>rotor</td>
<td>$C_{p_{max}}$</td>
</tr>
<tr>
<td>gearbox</td>
<td>$n_{tr}$</td>
</tr>
<tr>
<td>wind regime</td>
<td>$V_{in}$, $V_r$</td>
</tr>
</tbody>
</table>
Fig. 8.3 Typical example of a power output curve of a wind turbine, in this case of a Swedish 60 kW machine. The interval time chosen to measure the output and the wind speed was 10 minutes, according to international standards.
The starting wind speed $V_{\text{start}}$ is the wind speed at which the rotor starts turning, i.e. at that speed the rotor can overcome the starting torque of the generator and the gearbox. At $V_{\text{in}}$, however, the rotor produces already enough power to cover $P_{\text{mech}}(n_{\text{in}})$ and starts to produce nett power.

In designing a rotor to drive a generator, one has to bear in mind that the choice of the tip speed ratio is not as wide as it seems. Most often a two or three-bladed rotor will be chosen, so the tip speed ratios will probably vary between 5 and 8. Consequently, an easy, but lengthy, solution would be to repeat the procedure of section 8.1 a number of times for tip speed ratios of 5, 6, 7 and 8 for example. For each of the rotors a proper transmission ratio must be chosen and afterwards the resulting $P(V)$ curves can be judged by their values of $V_{\text{in}}$ and $V_{\text{r}}$.

Instead of this trial-and-error method, the following and more direct procedure is applicable to generators of which the speed is allowed to vary along with the wind speed. This means that one aims at keeping the mechanical power required to drive the generator (+ gearbox) as close as possible to the maximum power delivered by the wind rotor. This is particularly true at low wind speeds, leading to the assumption that $C_p \approx C_{p_{\text{max}}} \text{ at } V_{\text{in}}$.

The first indication whether a constant tip speed ratio ($\lambda_d$) can be maintained, is found by realizing that in that case the rotor speed, and also the generator speed, must at least be proportional to the wind speed:

$$n_r = n_{\text{in}} \frac{V_{r}}{V_{\text{in}}} \quad (8.1)$$

If $n_r$ is smaller than the value given by (8.1), then the tip speed ratio cannot remain constant and one has to apply another method. If $n_r$ is much higher, then $P_r$ will probably not be reached at $V_r$ so one of these choices was wrong. Here we assume that $n_r$ has the proper value.
First the necessary rotor area $A$ is determined with:

$$C_{p_{\text{max}}} \eta_{\text{tr}} \frac{1}{2} \rho A V_{\text{in}}^3 = P_{\text{mech}} (n_{\text{in}})$$  \hspace{1cm} (8.2)$$

When the rotor area is found one must check whether the rated power $P_r$ really can be produced at the required rated speed:

$$C_{p_{\text{max}}} \eta_{\text{tr}} \eta_{G} \frac{1}{2} \rho A V_r^3 > P_r$$  \hspace{1cm} (8.3)$$

If this condition cannot be fulfilled, one has to increase the rotor area accordingly or accept a higher value for $V_r$.

Subsequently, the relation between $\lambda_d$ and $i$ can be found with the assumption that $C_p = C_{p_{\text{max}}}$ at $V_{\text{in}}$ ($n_{\text{in}}$ is given in revolutions per second):

$$\lambda_d i = \frac{2\pi n_{\text{in}} R}{V_{\text{in}}}$$  \hspace{1cm} (8.4)$$

Basically any combination of tip speed ratio and $i$ can now be chosen. One restriction is however that the higher the tip speed ratio, the lower the starting torque of the rotor. We must ensure that the rotor can start at a wind speed $V_{\text{start}}$ lower than $V_{\text{in}}$. The starting torque is found with the help of the following empirical expression (see section 4.2):

$$C_{Q_{\text{start}}} = 0.5 \frac{\lambda_d^2}{\rho}$$  \hspace{1cm} (8.5)$$

This leads to:

$$0.5 \frac{\lambda_d^2}{\rho} V_{\text{start}}^2 A R = Q_{\text{start}} i$$  \hspace{1cm} (8.6)$$
Here we have neglected the starting torque of the gearbox, because it is generally much lower than the product of generator starting torque and $i$.

Realizing that $V_{\text{start}} < V_{\text{in}}$ we can rewrite (8.6) as follows:

$$\lambda_d^2 \lambda_1 < \frac{0.5 \lambda_0 V_{\text{in}}^2 A R}{Q_{\text{start}}}$$

(8.7)

Combining (8.7) with (8.4) yields:

$$\lambda_d < \frac{0.5 \lambda_0 V_{\text{in}}^2 A}{2\pi n_{\text{in}} Q_{\text{start}}}$$

(8.8)

8.3. Calculation example

We shall illustrate the procedure of section 8.2 with the following calculation example.

Assume a generator with the following characteristics:

$$P_r = 2000 \text{ W} \quad Q_{\text{start}} = 0.6 \text{ Nm}$$

$$n_r = 30 \text{ r.p.s.} \quad n_{\text{in}} = 10 \text{ r.p.s.}$$

$$n(n_r) = 0.8$$

$$P_{\text{mech}}(n_{\text{in}}) = 150 \text{ W}$$

The other necessary data are:

- gearbox: $n_{\text{tr}} = 0.9$
- rotor: $C_{P_{\text{max}}} = 0.35$
- windmill: $V_{\text{in}} = 4 \text{ m/s}$
- $V_r = 11 \text{ m/s}$. 
To find a suitable rotor we first check the rated speed with (8.1):

\[ 30 > 10 \times \frac{11}{4} \]

\[ 30 > 27.5 \quad \text{conclusion: O.K.} \]

Then the necessary rotor area is found with (8.2):

\[ A = \frac{150}{0.35 \times 0.9 \times 0.6 \times 4^3} \]

\[ A = 12.4 \, \text{m}^2 \]

or:

\[ R = 2 \, \text{m} \]

Now we check whether the rated power can be produced with this rotor area with the expression (8.3):

\[ 0.35 \times 0.9 \times 0.8 \times 0.6 \times 12.4 \times 11^3 = 2495 \, \text{W} \]

This is more than the 2000 W required, so the condition is fulfilled.

The relation between tip speed ratio and gear ratio \( i \) becomes (8.4):

\[ \lambda_d \times i = \frac{2 \times 3.14 \times 10 \times 2}{4} = 31.4 \]

The maximum allowable value for the tip speed ratio is found with (8.8):

\[ \lambda_d < \frac{0.5 \times 0.6 \times 4^3 \times 12.4}{2 \times 3.14 \times 10 \times 0.6} \]

\[ \lambda_d < 6.3 \]

We may conclude that, if one chooses to apply a three-bladed rotor for which a tip speed ratio of 6 is reasonable, the rotor will start below \( V_n \) and the necessary transmission ratio becomes:

\[ i = \frac{31.4}{6} = 5.2 \]

The resulting \( P(V) \) curve of the system can be found by the procedure described in section 8.1.
8.4. Mathematical description wind turbine output

The general expression for the (nett) power output of a wind turbine, as a function of the instantaneous (or usually short time average) wind speed is:

\[ P = C_p \eta \frac{1}{2} \rho V^3 A \]  \hspace{1cm} (8.9)

In this section we shall limit ourselves to a description of the output for wind speeds below the rated speed \( V_r \). It is assumed that all wind turbines limit their output power to a constant output \( P_r \) for wind velocities above \( V_r \) and below \( V_{\text{out}} \). For wind speeds above \( V_{\text{out}} \) the machine is shut down: \( P = 0 \). These assumptions imply that \( C_p \eta \) becomes proportional to \( 1/V^3 \) for \( V_r < V < V_{\text{out}} \) and zero for \( V > V_{\text{out}} \).

In the ideal case the factor \( C_p \eta \) should be equal to its highest value \( (C_p \eta)_{\text{max}} \) for all wind speeds \( V < V_r \):

**IDEAL:** \[ P = (C_\eta)_{\text{max}} \frac{1}{2} \rho V^3 A \]  \hspace{1cm} (8.10)

or: \[ P = \text{constant} \times V^3 \]

This is indeed the ideal for every wind turbine designer and, within his economic and structural constraints, he will try to approach this ideal cubic behaviour as closely as possible. In practice the result will be less ideal. First a practical wind turbine will only produce nett power above a given wind speed \( V_{\text{in}} \). Then the shape of the output curve between \( V_{\text{in}} \) and \( V_r \) may take any shape: linear, quadratic, cubic, even higher powers and combinations of these (cf the waterpumping windmill in section 6.2). We will describe a few of them, with their peculiarities (fig. 8.4).
The ideal output curve and two typical output curves of wind turbines.

Broadly the output curves can be divided into two groups:

1) Those reaching their highest efficiency between \( V_{in} \) and \( V_r \) i.e. the cubic curve \( (C_n)^{3/2} \) touches the output curve in a point with \( V_{in} < V_d < V_r \). This is the case with the linear output characteristic for example (fig. 8.4).

2) Those reaching their highest efficiency at \( V_r \), or in other words \( V_d = V_r \). This implies a steep output curve, such as shown in the third figure of fig. 8.4.

With the mathematical expressions for the output power as a function of windspeed, usually measured in the field, as shown in fig. 8.3, one can derive the expression for the design speed, the efficiency and the rated power for different rated speeds. This will be treated in detail for the linear output model, because the linear model usually gives a good match with the experimental data.
The linear behaviour can be written as:

\[
\text{LINEAR: } P = P_r \frac{V - V_{in}}{V_{r} - V_{in}} \quad (8.11)
\]

With (8.9) this becomes:

\[
C_p \beta_p V^3 A = (C_p \eta)_{\infty} V^3 A * \frac{V - V_{in}}{V_{r} - V_{in}} \quad (8.12)
\]

so we can find the \( C_p \eta \) at each velocity via:

\[
C_p \eta = (C_p \eta)_{\infty} * \frac{V^3}{V_{r} - V_{in}} * \frac{V - V_{in}}{V_{r} - V_{in}} \quad (8.13)
\]

If we want to determine the speed \( V_d \), at which by definition the maximum value of \( C_p \eta \) is reached, we have to take the derivative of (8.13):

\[
\frac{dC_p \eta}{dV} = (C_p \eta)_{\infty} \frac{V^3}{V_{r} - V_{in}} * (- \frac{2V_{in}}{V^3} + \frac{3V_{in}}{V^4}) \quad (8.14)
\]

Equating the derivative to zero for \( V = V_d \) gives:

\[
V_d = 1.5 V_{in} \quad (8.15)
\]

This means that for any wind turbine with a linear output characteristic the design speed is fixed and equal to 1.5 times the cut-in speed \( V_{in} \). Substituting this result in (8.13) and combining the result with (8.13) itself gives us an expression for \( C_p \eta \):

\[
C_p \eta = (C_p \eta)_{\max} \frac{V^3}{V_{d}^3} (3 \frac{V}{V_d} - 2) \quad (8.16)
\]
or, in terms of $V_{\text{in}}$:

$$C_{p, n} = (C_{p, n})_{\text{max}} \times 6.75 \frac{V_{\text{in}}^3}{V_{\text{in}}^3} \left(\frac{V_{\text{in}}}{V_{\text{in}} - 1}\right)$$  \hspace{1cm} (8.17)

The expression (8.16) is shown in fig. 8.5.

To find the rated power for different rated wind speeds $V_r$ is substituted in (8.17) and multiples both members of the equation with $\frac{1}{3}p A V^3$. The result is:

$$P_r = (C_{p, n})_{\text{max}} \frac{1}{3}p A V^3 (3 \frac{V_r}{V_d} - 2)$$  \hspace{1cm} (8.18)

A similar procedure can be followed for a general output model (cf ref. 26):

$$P = \frac{\frac{c}{V - V_{\text{in}}}}{V_r - V_{\text{in}}} \frac{V_r}{V_{\text{in}}}$$  \hspace{1cm} (8.19)

With (8.9) this becomes:

$$C_{p, n} = (C_{p, n})_r \frac{V_r^3}{V_r^3} \left(V_{c, \text{in}} - V_{c, \text{in}}\right)$$  \hspace{1cm} (8.20)

Taking the derivative to $V$ equal to zero gives the design speed:

$$V_d = V_{\text{in}} - c \sqrt{\frac{3}{3-c}}$$  \hspace{1cm} (8.21)

For $c=1$ we find the result of equation (8.15). We also see that for $c > 3$ no design speed is found, simply because the output curve is steeper than the ideal cubic output curve that will intersect the output curve now only at $V_r = V_d$ by definition.
The efficiency is calculated to be:

\[
C_p \eta = (C_p \eta)_{\text{max}} \left[ 1 - \frac{3}{c} \left( 1 - \frac{c}{3} \right) \right] \left( 1 - \frac{3}{c} \right) \left( \frac{V_{\text{in}}}{V_c} \right)^3 \left( \frac{V_c}{V_{\text{in}}} - 1 \right)
\]  

(8.22)

and, as expected, for \( c=1 \) we find expression (8.17).

The rated power is given by:

\[
P_r = \frac{3}{c} \left( 1 - \frac{c}{3} \right) \left( \frac{V_{\text{in}}}{V_c} \right)^3 \left( \frac{V_c}{V_{\text{in}}} - 1 \right)
\]  

(8.23)

We shall use these models in section 9.2 to calculate the output of a wind turbine in a Weibull distributed wind regime.
Fig. 8.5 The relative efficiency of a wind turbine as a function of wind speed, where the wind speed is made dimensionless by dividing by the design wind speed $V_d$. The efficiency curve is based upon a linear output versus wind speed characteristic of the wind turbine between $V_{in}$ and $V_r$. 
9. MATCHING WINDMILLS TO WIND REGIMES: OUTPUT AND AVAILABILITY

Now that we possess both the power output curves of windmills (chapters 6 and 8) and the wind velocity distribution of the wind regime (chapter 3) we can combine them to calculate the output of a windmill and also the availability of the output.

The availability of the output comes into the picture because a high output nearly always implicates a low availability. The reason is that the high output is attained by designing the windmill specifically for high wind speeds, with the result that the machine will hardly ever operate at low wind speeds which are usually more frequent. So in the matching process one has to compromise between output and availability.

9.1 Outline of the methods

Apart from the simple rule of thumb of chapter 2 (\(E = 0.1 AV^3T\)) we will discuss three methods here: a graphical method, a computation method and an estimation method based upon a mathematical approximation of the wind velocity distribution.

9.1.1 Graphical approach

Basically this method consists of transforming the velocity duration curve of the wind regime into a power duration curve (see fig. 9.1). This is done by indicating \(V_{in}\), \(V_r\) and \(V_{out}\) in the velocity duration curve, finding the corresponding time fractions and transferring them to the power duration curve.

The energy output is given by the area under the power duration curve. This method gives a good insight in the effects of changing the rated speed or the cut-in speed of the windmill. Also the availability is shown directly on the horizontal axis, as the time fraction that power is being produced. The main drawback is that the energy output itself has to be found by a graphical integration of the power duration curve which is not very convenient.
Fig. 9.1 Graphical method to find the energy output of a windmill by means of the velocity duration curve.

9.1.2 Computational approach

The computation method utilizes the velocity frequency distribution and consists of multiplying the number of hours in each wind speed interval with the corresponding power output. The sum of all these products gives the energy output (fig. 9.2).
We shall illustrate the method with the following example. We assume that a $\phi 3$ m water pumping windmill with a linear output characteristic is installed in Hambantota (Sri Lanka) with an average wind speed of 5.6 m/s. We choose a design wind speed equal to the average wind speed: $V_d = 5.6$ m/s and $(C_n)_{\text{max}} = 0.2$. The other characteristic speeds are: $V_{\text{in}} = 3.7$ m/s ($V_d/1.5$), $V_r = 8$ m/s and $V_{\text{out}} = 12$ m/s. The water output is calculated for a total head of 10 m. The results are given below.

<table>
<thead>
<tr>
<th>windspeed interval (m/s)</th>
<th>frequency distribution</th>
<th>nett power of the windmill (W)</th>
<th>nett energy of the windmill (kWh)</th>
<th>water output at H = 10 m (m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 1</td>
<td>285</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1 - 2</td>
<td>733</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2 - 3</td>
<td>945</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3 - 4</td>
<td>1088</td>
<td>7</td>
<td>7.6</td>
<td>279</td>
</tr>
<tr>
<td>4 - 5</td>
<td>1193</td>
<td>63</td>
<td>74.8</td>
<td>2745</td>
</tr>
<tr>
<td>5 - 6</td>
<td>1127</td>
<td>141</td>
<td>159.0</td>
<td>5835</td>
</tr>
<tr>
<td>6 - 7</td>
<td>891</td>
<td>219</td>
<td>195.5</td>
<td>7175</td>
</tr>
<tr>
<td>7 - 8</td>
<td>722</td>
<td>298</td>
<td>215.0</td>
<td>7890</td>
</tr>
<tr>
<td>8 - 9</td>
<td>556</td>
<td>337</td>
<td>187.4</td>
<td>6878</td>
</tr>
<tr>
<td>9 - 10</td>
<td>377</td>
<td>337</td>
<td>127.0</td>
<td>4661</td>
</tr>
<tr>
<td>10 - 11</td>
<td>297</td>
<td>337</td>
<td>100.0</td>
<td>3670</td>
</tr>
<tr>
<td>11 - 12</td>
<td>205</td>
<td>337</td>
<td>69.1</td>
<td>2536</td>
</tr>
<tr>
<td>12 - 13</td>
<td>113</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13 - 14</td>
<td>106</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14 - 15</td>
<td>43</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>TOTAL</td>
<td></td>
<td>1135.4 kWh</td>
<td>41669 m$^3$</td>
<td></td>
</tr>
</tbody>
</table>

The number of hours that the windmill operates is found by addition of the hours in the corresponding intervals. This results in 5368 and so the availability is 5368/8760 or 61.3%. The windmill is not operating due to low wind speeds during 3051 hours (34.8% of the year) and during 341 hours the wind speed is above $V_{\text{out}}$ (3.9%).
9.1.3 Estimation method

The estimation method is basically identical to the computation method, but utilizes mathematical approximations for the velocity frequency curve and for the output curve of the windmill. The formulas involved will be discussed in section 9.2 and here we shall only use the results of the method.

In order to arrive at universally applicable values, all wind speeds are made dimensionless by dividing them by the average wind speed.

\[ x = \frac{V}{\bar{V}} \]  

(9.1)

Also the energy output is given in a dimensionless form by introducing:

\[ e_{\text{system}} = \frac{E}{(C_p \eta)_{\text{max}} \cdot \rho \cdot A \cdot \bar{V}^3 \cdot T} \]  

(9.2)

Because the design speed \( V_d \) of the windmill is a key parameter in the matching process, the dimensionless energy output is plotted versus \( x_d = V_d / \bar{V} \) in fig. 9.3 for different values of \( x_r \) assuming \( x_{out} = \infty \). In fig. 9.3 this is done for a given velocity frequency distribution applicable to coastal areas like Hambantota, and indicated by a so-called Weibull shape factor \( k = 2 \). The Weibull approximation will be analyzed further in section 9.2.

Now we can compare this estimation with the second method. With the given windmill characteristics we see that \( x_d = 1 \) and \( x_r = 1.43 \). Applying these values to fig. 9.3 results in \( e_{\text{system}} = 0.99 \).

The annual output now is found with (fig. 9.2):

\[ E = 0.99 \times 0.2 \times 0.6 \times \pi \times 1.5^2 \times 5.6^3 \times 8760 \]

\[ E = 1292 \text{ kWh/year} \]
This output is too high, however, because in fig. 9.3 it is assumed that \( x_{\text{out}} = \infty \), whereas in our case \( x_{\text{out}} = \frac{12}{5.6} = 2.14 \). The effect of this cut-out wind speed can be seen in fig. 9.4, and we conclude that \( e_{\text{system}} \) should be decreased with about 0.07. This leads to:

\[
E = 1200 \text{kWh/year}
\]

or

\[
E = 44,000 \text{m}^3/\text{year over 10 m head}.
\]

It is interesting to compare the outputs of the second and the third method with the simple rule of thumb which we introduced in chapter 2:

\[
E = 0.1 \pi \times 1.5^2 \times 5.6^3 \times 8760
\]

or

\[
E = 1087 \text{kWh/year}
\]

or

\[
E = 39900 \text{m}^3/\text{year over 10 m head}.
\]

Please note that the latter value is quite close only because we choose reasonable values for \( (C_p n)_{\text{max}} \) and for \( x_d \) and \( x_r \). For other values the outcome can be quite different.

We conclude that the most exact output is given by the computational approach. But, and this is important to realize, we have used the wind data of one year only. Much better is to use the average of a ten year period and even in that case we must realize that the deviation from this average might easily be 10% or more next year and that is the year for which we want to predict the output. Taken all these factors into account, the estimation approach with the dimensionless energy output gives us very quickly rather accurate output values, although only for windmills with a linear output (for other output curves other graphs must be drawn).
Fig. 9.3 The dimensionless energy output of windmills with a linear output curve and $V_{out} = \ldots$ in a Weibull wind regime with $k = 2$. 
Fig. 9.4  The effect of the cut-out wind speed on the dimensionless energy output of windmills with a linear output curve, in a Weibull wind regime with $k = 2$. 
9.2 Mathematical description of output and availability

In analogy with the computational approach of section 9.1.2, the output of a wind turbine, with a power-wind speed curve $P(V)$ in a wind regime with a frequency distribution $f(V)$ in a period $T$ can be written as:

$$ E = T \int_{0}^{\infty} P(V) f(V) \, dV $$

(9.3)

Usually the power-wind speed curve takes one of the shapes shown in fig. 8.4, i.e. increasing from 0 to $P_r$ between $V_{in}$ and $V_r$ and equal to $P_r$ between $V_r$ and $V_{out}$.

This changes (9.3) into:

$$ E = T \int_{V_{in}}^{V_r} P(V) f(V) \, dV + T P_r \int_{V_r}^{V_{out}} f(V) \, dV $$

(9.4)

Using the dimensionless wind speeds $x = \frac{V}{V}$ one obtains:

$$ E = T \int_{x_{in}}^{x_r} P(x) f(x) \, dx + T P_r \int_{x_r}^{x_{out}} f(x) \, dx $$

(9.4)

The dimensionless energy output is defined by:

$$ e_{system} = \frac{E}{(C_p \eta)_{max} \frac{1}{2} A \overline{V}^3 T} = \frac{E}{P_{ref} T} $$

(9.6)

with (9.6) the expression (9.5) changes into:

$$ e_{system} = \frac{1}{P_{ref}} \int_{x_{in}}^{x_r} P(x) f(x) \, dx + \frac{P_r}{P_{ref}} \int_{x_r}^{x_{out}} f(x) \, dx $$

(9.7)
The expression (9.4) will be worked out for a number of power-wind speed curves $P(V)$, assuming a Weibull distribution for $f(V)$. The integrals found can be solved numerically with Simpson's rule (currently available on many pocket calculators).

9.2.1 Ideal wind turbine

The ideal wind turbine possesses a cubic $P(V)$ curve for $V < V_r$:

$$P(V) = (C_n \eta)_{\max} \frac{1}{\rho} A V^3$$  \hspace{1cm} (9.8)

so

$$P(x) = (C_n \eta)_{\max} \frac{1}{\rho} A x^3 \frac{V^3}{V_r^3} = P_{\text{ref}} x^3$$  \hspace{1cm} (9.9)

With the expression for $f(x)$ (see section 3.4.2) and with:

$$G = \Gamma^k \left(1 + \frac{1}{k}\right)$$  \hspace{1cm} (9.10)

the dimensionless energy output becomes:

$$e_{\text{system}} = G \int_{x_{\text{ref}}}^{x_r} x^{k+2} e^{-Gx^k} - Gx^k_{\text{ref}} - Gx^k_{\text{out}} \, dx + x_r^3 \left(e^{-\Gamma x^k_{\text{out}}} - e^{-\Gamma x^k_{\text{ref}}} \right)$$  \hspace{1cm} (9.11)

This expression is shown in fig. 9.5 for different values of $k$, assuming $x_{\text{in}} = 0$ and $x_{\text{out}} = \infty$.

Note that for $k = 2$ the value of the gamma function is $G = \frac{\pi}{4}$, thereby transforming (9.11) into:

$$e_{\text{system}} = \frac{\pi}{2} \int_{x_{\text{in}}}^{x_r} x^4 e^{-\frac{\pi}{4} x^2} - \frac{\pi}{4} x_r^2 - \frac{\pi}{4} x_{\text{out}}^2 - \frac{\pi}{4} x_{\text{ref}}^2 \, dx + x_r^3 \left(e^{-\frac{\pi}{4} x_{\text{out}}^2} - e^{-\frac{\pi}{4} x_{\text{ref}}^2} \right)$$  \hspace{1cm} (9.12)
The values of $e_{\text{system}}$ given in fig. 9.5 represent the absolute maximum energy that can be extracted in a given wind regime and for a given value of $x_r = V_r/\bar{V}$. The maximum extractable amount of energy is defined by the energy pattern factor $k_E$ (see section 3.4.3) and in fig. 9.5 it is shown that the maximum value of $e_{\text{system}}$ approaches the value of $k_E$ for a given Weibull shape factor $k$.

For example: for $k = 2$ the energy pattern factor is $k_E = 1.9$.

### 9.2.2 Wind turbine with linear output curve

The linear output between $V_{\text{in}}$ and $V_r$ is given by:

$$P = P_r \frac{V - V_{\text{in}}}{V_r - V_{\text{in}}}$$  \hspace{1cm} (9.13)

Integration by parts and rearranging yields:

$$E = \frac{T P_r}{x_r - x_{\text{in}}} \int_{x_{\text{in}}}^{x_r} e^{-Gx^k} dx - T P_r e^{-Gx^k}$$  \hspace{1cm} (9.14)

In order to find the dimensionless energy output, the rated output $P_r$ has to be expressed in $(C_p \eta)^{\text{max}}$, $V_{\text{in}}$ and $V_r$ with the help of expression (8.17) for $V = V_r$:

$$P_r = (C_p \eta)^{\text{max}} 6.75 \frac{V_{\text{in}}^3}{V_r^3} (\frac{V_r}{V_{\text{in}}} - 1) \frac{1}{2} \rho A \bar{V}^3$$  \hspace{1cm} (9.15)

Dividing by the expression for $P_{\text{ref}}$:

$$P_{\text{ref}} = (C_p \eta)^{\text{max}} \frac{1}{2} \rho A \bar{V}^3$$

This gives:
\[ p_{\text{ref}} = 6.75 x_{\text{in}}^2 (x_r - x_{\text{in}}) \] (9.16)

The resulting dimensionless energy output becomes:

\[ e_{\text{system}} = 6.75 x_{\text{in}}^2 \int_0^{x_r} e^{-Gx_k} dx - 6.75 x_{\text{in}}^2 (x_r - x_{\text{in}}) e^{-Gx_{\text{out}}} \] (9.17)

In terms of the design speed \( x_d \) we can write (see formula 8.16):

\[ e_{\text{system}} = 3x_d^2 \frac{x_r}{2} e^{-Gx_k} dx - x_d^2 (3x_r - 2x_d) e^{-Gx_{\text{out}}} \] (9.18)

The latter function has been shown in figs. 9.3 and 9.4 for a Weibull shape factor \( k = 2 \). One realises now that the function in the third quadrant of fig. 9.4 is simply \( x_d^2 (3x_r - 2x_d) \), the value of which is multiplied by \( \exp(-Gx_{\text{out}}) \) in the first quadrant to find the correction value \( \Delta e_{\text{system}} \).

9.2.3 Constant torque load

The power output of an ideal water pumping windmill, consisting of a rotor with a linear \( C_Q - \lambda \) curve and a pump with a constant torque, is given by (see section 6.2):

\[ P = \left[ \frac{V}{V_d} L - \frac{V_d}{V} (L - 1) \right] (C_p n)_{\text{max}} \frac{1}{2} \rho A V_d^3 \] (9.19)

with \( L = \frac{\lambda_{\max}}{\lambda_d} \)
The cut-in speed of this ideal windmill is given by

\[ V_{in} = V_d \sqrt{1 - \frac{1}{L}} \]  

(9.20)

The dimensionless energy output becomes:

\[ e_{system} = x_d^3 \frac{x_r}{G_k} \int \left[ \frac{x}{L} - \frac{x_d}{x} \right] (L - 1) e^{k-1 - Gx^k} + \]

\[ x_d^3 \left[ \frac{x_r}{x_d} - \frac{x^2}{x_r} \right] (e^{-Gx^k} - e^{-Gx^k_{out}}) \]  

(9.21)

Two examples of this expression in figs. 9.6 and 9.7, the first for \( x_d = 1.4 \) and the second for \( x_d = 1 \).

9.2.4 Wind turbine with quadratic \( P(V) \) curve

The quadratic \( P(V) \) curve chosen is:

\[ P(V) = P_r \frac{v^2 - v_{in}^2}{v_{r}^2 - v_{in}^2} \]  

(9.22)

This is a special case of a general formula introduced by Powell [26]:

\[ P(V) = P_r \frac{v^k - v_{in}^k}{v_{r}^k - v_{in}^k} \]  

(9.23)
This formula was chosen because with this expression the integral (9.4) can be solved, so an analytical expression for the energy output results. The general formula (9.23) is rather questionable however, because it suggests that $P(V)$ is a function of the Weibull factor $k$, which is not the case. Nevertheless we shall work out the integral (9.4) and later on limit ourselves to the $k = 2$ case.

Writing the integral with dimensionless wind speeds:

$$E = \frac{T P_r}{k} \int_{x_r}^{x_{\text{out}}} \left( x^k - x_{\text{in}}^k \right) G k x^{k-1} e^{-G x^k} \, dx +$$

$$T P_r \int_{x_r}^{x_{\text{out}}} G k x^{k-1} e^{-G x^k} \, dx \tag{9.24}$$

and realising that:

$$d e^{-G x^k} = -G k x^{k-1} e^{-G x^k} \, dx \tag{9.25}$$

the integral can be reduced to:

$$E = \frac{T P_r}{k} \left[ \int_{x_r}^{x_{\text{out}}} k \, e^{-G x^k} - x_{\text{in}}^k (e^{-G x_{\text{in}}^k} - e^{-G x_r^k}) \right] +$$

$$T P_r \left( e^{-G x_{\text{out}}^k} - e^{-G x_r^k} \right) \tag{9.26}$$

The remaining integration is solved via integration by parts:

$$- \int_{x_{\text{in}}}^{x_r} k \, e^{-G x^k} = - (x_r^k e^{-G x_{\text{out}}^k} - x_{\text{in}}^k e^{-G x_{\text{in}}^k}) + \frac{1}{G} \int_{x_{\text{in}}}^{x_r} e^{-G x^k} \, dx \tag{9.27}$$
This reduces to:
\[ e^{-Gx_k} (x_{\text{in}}^k + \frac{1}{G}) - e^{-Gx_r} (x_{\text{r}}^k + \frac{1}{G}) \] (9.28)

Substituting (9.28) into (9.26) yields:
\[ E = \frac{T P_r}{G(x_{\text{r}} - x_{\text{in}})} (-Gx_{\text{in}}^k - e^{-Gx_{\text{r}}^k} - T P_r e^{-Gx_{\text{out}}^k}) \] (9.29)

In order to find an expression for the dimensionless energy output \( e \), the rated output \( P_r \) has to be expressed in terms of \( (C_p n)_m^\text{max}, V_{\text{in}} \) and \( V_{\text{r}} \).

Generally we can state:
\[ P = C_{\text{p}} n \frac{1}{2} \rho A V^3 \] (9.30)

This is also true for \( P_r \):
\[ P_r = (C_{\text{p}} n)_r \frac{1}{2} \rho A V^3 \] (9.31)

Combining these two expressions with the formula (9.23) for \( P(V) \) we arrive at:
\[ C_{\text{p}} n \frac{1}{2} \rho A V^3 = (C_{\text{p}} n)_r \frac{1}{2} \rho A V^3 \frac{v^k - v_{\text{in}}^k}{v_{\text{r}}^k - v_{\text{in}}^k} \] (9.32)
or:
\[ C_{\text{p}} n = (C_{\text{p}} n)_r \frac{v^3 v^k - v_{\text{in}}^k}{v_{\text{r}}^3 v^k - v_{\text{in}}^k} \] (9.33)
Taking the derivative of (9.33) with respect to \( V \) to find the maximum value of \( C_n \), i.e. \( (C_n)_\text{max} \), we find that this maximum occurs for a (design) speed:

\[
V_d = V_{\text{in}} \sqrt{\frac{3}{3-k}}
\]  

(9.34)

Substituting (9.34) into (9.33) yields:

\[
(C_n)_V = (C_n)\max \frac{3}{k} (1 - \frac{k}{3}) \left[ 1 - \frac{3}{k} \frac{V_{\text{in}}^{3-k}}{V_r^{3-k}} (V_r - V_{\text{in}}^k) \right]
\]  

(9.35)

and

\[
P_r = (C_n)\max \frac{3}{k} (1 - \frac{k}{3}) \left[ 1 - \frac{3}{k} \frac{V_{\text{in}}^{3-k}}{V_r^{3-k}} (V_r - V_{\text{in}}^k) \right]
\]  

(9.36)

Substitution in 9.29 and dividing by \( P_{\text{ref}} T \) (see 9.6) gives:

\[
e_{\text{system}} = \frac{3}{k} (1 - \frac{k}{3}) \left[ \frac{3-k}{6} \frac{1}{G} (e^{G_{\text{in}}^k} - e^{G_{\text{r}}^k}) - \frac{G_x^k}{(x_r^k - x_{\text{in}}^k)} e^{-G_x^k} \right]
\]  

(9.37)

For \( k = 2 \) this reduces to:

\[
e_{\text{system}} = 1.5 \sqrt{3} x_{\text{in}} \left[ \frac{4}{\pi} (e^{-\frac{\pi}{4} x_{\text{in}}^2} - e^{-\frac{\pi}{4} x_r^2}) - \frac{\pi}{4} x_{\text{out}}^2 (x_r^2 - x_{\text{in}}^2) e^{-\frac{\pi}{4} x_{\text{out}}^2} \right]
\]  

(9.38)

This expression is also shown in figs. 9.6 and 9.7 and indicated by \( \sim v^2 \).
9.2.5 Comparison of the four P(V) curves

For comparison purposes the four P(V) curves mentioned before are plotted for two situations:

(I) \( k = 2, x_d = 1.4 \) and \( x_{out} = \infty \)

(II) \( k = 2, x_d = 1.0 \) and \( x_{out} = \infty \)

The design speed \( x_d = 1.4 \) has been chosen because the three non-cubic P(V) curves produce their highest output at, roughly, this value. The design speed of \( x_d = 1.0 \) is a more practical value, resulting in a slightly lower output but a higher availability of the power.

9.2.6 Availability of power

The availability \( \tau \) of the power from a wind turbine is defined as the fraction of the total time at which the wind speed is sufficient to operate the machine. The turbine cannot operate during all hours with relative speeds between 0 and \( x_{in} \) and with relative speeds higher than \( x_{out} \):

\[
1 - \tau = \int_0^{x_{in}} f(x) \, dx + \int_{x_{out}}^{\infty} f(x) \, dx
\]

(9.39)

By definition, these terms represent the cumulative distribution function \( F(x) \):

\[
1 - \tau = F(x_{in}) + 1 - F(x_{out}) \quad \text{and} \quad \tau = F(x_{out}) - F(x_{in})
\]

(9.40)
Their value can be found in fig. 3.9 and in formula the expression becomes:

\[
\tau = (1 - e^{-G_{\text{out}}}) - (1 - e^{-G_{\text{in}}})
\]

or:

\[
\tau = e^{-G_{\text{in}}} - e^{-G_{\text{out}}}
\]

(9.41)
Fig. 9.5 The value of $e_{\text{system}}$ as a function of the rated relative velocity $x_r$ for several values of the Weibull shape factor (related to a wind turbine with an ideal output characteristic).
Fig. 9.6 The value of $e_{\text{system}}$ as a function of the rated relative velocity $x_r$ for four different types of power wind speed curves. The value of $x_d$ is 1.4.
Fig. 9.7 The value of $e_{\text{system}}$ as a function of the rated relative velocity $x_r$ for four different types of power wind speed curves. The value of $x_d$ is 1.
10. **ROTOR STRESS CALCULATIONS**

10.1 General

A windmill operates by virtue of the forces that the wind exerts on its rotor. These forces are transferred to the load (pump or generator) and the interplay between rotor and load causes various moments and forces in each part of the windmill structure. Some of them are listed below, but the list is not exhaustive.

- **Blades**: torque, thrust, weight (inertia), forces imposed by load
- **Shaft**: torsion, bending (pump rod forces and gyroscopic moment)
- **Shaft bearings**: axial force (thrust), radial force (pump rod force, weight of rotor and gyroscopic moment)
- **Pump rod**: tensile forces, compression forces (buckling)
- **Head bearing**: radial force (thrust on rotor), axial force (weight of head, pump rod force)
- **Vane**: aerodynamic forces, weight
- **Tower**: compression forces (weight, wind), tensile forces (wind), moments (due to loads on head)

* This chapter is written by R. Schermerhorn, member of the Wind Energy Group, Eindhoven University of Technology.
In principle the stresses in any part of the structure can be calculated for the maximum gust speed, even down to each nut, each flange, each welding. In practice usually the most vital parts of the structure are analysed such as the blades, the bearings and the tower. In this chapter we will give a few examples of these calculations.

Since the highest loads occur at the hub of the rotor, the cross section of the blade at the hub is in most cases the critical cross section. Furthermore, the load on a blade depends on the position of the blade in the rotor plane.

This calculation on rotor loads is mainly intended for slow running rotors, as designed by SWD for water pumping windmills with steel rotor blades on steel spokes.

It is assumed that the reader is familiar with basic formulas for stress and strain, such as can be found in the literature [27, 28].

The load, coupled to the rotor, is a single acting piston pump which has a cyclic torque (see section 5.2).

In this calculation all components are assumed to be stiff; no elastic deformation of blades and shaft has been taken into account.

Furthermore it is assumed that the aerodynamic center of the blade coincides with the rotor spoke, hence the aerodynamic forces only impose a bending moment and no torsion on the spokes of the rotor blades.

Indices will be frequently used in this chapter.

The following system of indexing will be employed as much as possible:

\( X_{123} \) - the first index denotes the element of the windmill for which the property holds or on which a force or moment is acting.

Examples: \( I_r \) : inertia of the rotor
\( F_b \) : force on a blade
\( M_p \) : moment exerted by the pump
-2- the second index refers to the origin of the force or moment.

Examples: $F_{rt}$ : force on the rotor due to thrust

$M_{bq}$ : moment on a blade due to torque

-3- the third index denotes the type of force or moment, i.e.

axial, radial, tangential.

Examples: $F_{rit}$ : force on the rotor due to inertia, in a tangential direction.

$M_{bga}$ : moment on a blade due to weight (gravity) on the axial axis.

10.2 Loads on a rotor blade

The loads on a rotor blade arise from:

- Aerodynamic forces: torque and thrust.
- Mass forces: weight and inertia.
- Forces due to load: pump rod force.

In this section these three loads will be discussed.

10.2.1 Aerodynamic forces

The rotation of the rotor is caused by moments acting on the rotation axis of the rotor (rotor shaft).

These moments are due to aerodynamic forces on the rotor in the plane of the rotor (torque $Q_r$) and the moment $M_p$ due to the pump rod force $F_p$ (see fig. 10.1).
Fig. 10.1 Axial moments on the rotor.

Since the moments of inertia of rotor hub and rotor shaft are small with respect to the moments of inertia of the blades, the equation of motion of the rotor (neglecting friction) is:

\[ Q_r - M_p = I_r \frac{d^2 \theta}{dt^2} \]  \hfill (10.1)

in which \( I_r \) is the moment of inertia of the rotor.

Due to the cyclic character of \( M_p \) (see section 5.2) two situations have to be considered.

a. The piston is moving downward, the pump rod force is assumed to be zero and the rotor is accelerating. This situation is dealt with under "Torque".

b. The piston is moving upward, pumping water and the rotor is decelerating. This situation is dealt with in section 10.2.3.
TORQUE

The torque on the rotor is the resulting moment on the rotor axis of the assumed uniformly distributed aerodynamic forces on the blades.

The torque $Q_r$ can be calculated from:

$$Q_r = C_Q \frac{1}{2} \rho V^2 \pi R^3$$  \hspace{1cm} (10.2)

When neglecting the inertia of rotor shaft, crank and pump rod with respect to the inertia of the rotor, the torque imposes no moment on the rotor shaft and as a consequence no bending moment on the rotor spokes at the hub.

As is described in eq. (10.1), the rotor rotates due to torque and pump load. In this chapter the effect of torque on the loads of the rotor blades is determined, while the effect of pump load will be determined under 10.2.3. When only taking the effect of the torque into account, eq. 10.1 becomes:

$$Q_r = I_r \frac{d^2 \theta}{dt^2}$$  \hspace{1cm} (10.3)

and, with $I_r = BI_b$:

$$\frac{d^2 \theta}{dt^2} = \frac{Q_r}{BI_b}$$  \hspace{1cm} (10.4)

The equation of motion for a blade element $dm$ is described by the second Law of Newton (see fig. 10.2):

$$dF = r \frac{d^2 \theta}{dt^2} dm$$  \hspace{1cm} (10.5)
Fig. 10.2 Acceleration forces on a blade element of a rotor blade.

The shearing force \( F_{bq} \) on a blade (near the hub) is the resultant force of all acceleration forces \( dF \)

\[
F_{bq} = \int dF = \int r \frac{d^2 \theta}{dt^2} dm = \frac{d^2 \theta}{dt^2} \int r dm = \frac{d^2 \theta}{dt^2} J_b \tag{10.6}
\]

where \( J_b = \int r dm \), the first moment of inertia of one blade.

With (10.4) we obtain:

\[
F_{bq} = \frac{Q_r J_b}{BI_b} \tag{10.7}
\]
With (10.2) this can be written as:

\[ F_{bq} = C_q \frac{k_0}{\rho} V^2 \pi R^3 \frac{J_b}{BI_b} \]  

(10.8)

THRUST

The thrust \( F_{rt} \) is the resulting axial force on the rotor and arises from aerodynamical forces which are assumed to be linear with \( r \).

Thrust imposes a shearing force as well as a bending moment on a rotor blade at the hub.

The shearing force per blade \( F_{bt} \) can be calculated from:

\[ F_{bt} = C_t \frac{k_0}{\rho} V^2 \pi \frac{R^2}{B} \]  

(10.9)

The bending moment can be calculated from:

\[ M_{bt} = F_{bt} \frac{2}{3} R \]  

(10.10)

Substituting (10.9) in (10.10) yields:

\[ M_{bt} = \frac{1}{3} C_t \rho V^2 \pi \frac{R^3}{B} \]  

(10.11)

10.2.2 Mass forces

GRAVITY

The weight of a blade imposes a bending moment \( M_{bg} \), a tangential force \( F_{bgt} \) and a radial force \( F_{bgr} \) upon the rotor blade at the hub.
Using the co-ordinate system from fig. 10.3, forces and moment due to weight can be expressed as:

\[ M_{bga} = -mgL \sin \theta \]  
(10.12)

\[ F_{bgt} = -mg \sin \theta \]  
(10.13)

\[ F_{bgr} = mg \cos \theta \]  
(10.14)
INERTIA

Simultaneous yawing of the head of the rotor and turning of the rotor imposes inertia forces and moments on the rotor blades. A derivation of the formulas for these forces and moments is given in appendix B.

The forces and moments due to inertia, as derived in this appendix are listed hereunder:

Axial force

\[ F_{ia} = 2 \cos \theta \frac{\Omega}{x} \frac{\Omega}{z} J_b \]  \hspace{1cm} (10.15)

Tangential force

\[ F_{it} = \sin \theta \cos \theta \frac{\Omega^2}{z} J_b \]  \hspace{1cm} (10.16)

Radial force

\[ F_{ir} = \frac{\Omega^2}{x} J_b + \sin^2 \theta \frac{\Omega^2}{z} J_b \]  \hspace{1cm} (10.17)

Moment on the axial axis

\[ M_{bia} = \sin \theta \cos \theta \frac{\Omega^2}{z} I_b \]  \hspace{1cm} (10.18)

Moment on the tangential axis

\[ M_{bit} = 2 \cos \theta \frac{\Omega}{x} \frac{\Omega}{z} I_b \]  \hspace{1cm} (10.19)

10.2.3 Forces due to pump load

As is shown in section 5.2, a single-acting piston pump has a cyclic torque.

In 10.2.1 the forces on a rotor blade have been calculated for the case of the pump rod force being zero, i.e. in the downward stroke of the piston.

Hereunder, the loads on a rotor blade are calculated during the upward stroke of the piston, when \( M_p \neq 0 \). Neglecting the inertia of rotor shaft, crank and hub with respect to the inertia of the rotor blades, the pump rod moment \( M_p \) imposes a bending moment and a shearing force on the rotor blades.
The pump rod force $F_p$ arises from static, acceleration, friction, and shock loads of water and pump rod. Although being quite complex, an approximation of this force can be made, realising that work done by the wind during one cycle is equal to work done by the pump rod force.

The work done by the pump rod force is

$$
\int_0^{2\pi} F_p \frac{s}{2} \sin \theta \, d\theta = \int_0^{2\pi} M_p \, d\theta = \int_0^{2\pi} Q_r \, d\theta
$$

(10.20)

To solve this equation, $F_p = F_p(\theta)$ must be known. The assumption that $F_p$ is constant during the upward movement of the piston can be justified by realising that the largest deviation of $F_p$ from the constant average occurs at the opening and closing of the valves, i.e. for $\theta$ is approximately 0 and $\pi$.

The contribution of $F_p'$ for $\theta$ equal to 0 or $\pi$, to the total work done by $F_p(\theta)$ during one cycle is very small, because of the factor $\sin \theta$ in (10.20).

Hence the work $W_p$ done by a constant $F_p$ during one cycle is:

$$
W_p = \int_0^{2\pi} F_p \frac{s}{2} \sin \theta \, d\theta = F_p \, s
$$

(10.21)

The work $W_Q$ done by the torque $Q_r$, which is assumed to be constant during one cycle is:

$$
W_Q = \int_0^{2\pi} Q_r \, d\theta = 2\pi \, Q_r
$$

(10.22)

Since $W_p = W_Q$ we obtain:

$$
F_p = \frac{2\pi Q_r}{s}
$$

(10.23)
Since $F_p$ only acts when lifting the water, the forces and moments imposed on a blade depend on the position of the crank, denoted by the angle $(\theta + \theta_b)$, where $\theta_b$ is the angle between crank and blade in question.

The bending moment $M_{bp}$ on a rotor blade at the hub due to the pump rod moment $M_p$ is:

$$M_{bp} = -\frac{F_p s \sin(\theta + \theta_b)}{2B} =$$

$$= -C_{Q_{bp}} V^2 \pi^2 R^3 \frac{\sin (\theta + \theta_b)}{B} \quad 0 < \theta + \theta_b < \pi \quad (10.24)$$

$$M_{bp} = 0 \quad \pi < \theta + \theta_b < 2\pi$$

Due to the varying pump rod moment, shearing forces occur in the rotor spokes at the hub, in the plane of the rotor.

These shearing forces can be calculated, realising that they arise from acceleration forces on the blades.

The equation of motion of a blade element $dm$ (see fig. 10.2) is described by the second law of Newton.

By integrating the acceleration $dF$, the resultant shearing force $F_{bp}$ can be calculated and the result is similar to equation (10.6).

$$F_{bp} = \frac{d^2\theta}{dt^2} J_b \quad (10.25)$$

where $J_b = \int_0^R dm$ is the first moment of inertia of one blade.

The acceleration of the rotor is found with equation (10.1) in which $Q_c$ can be thought to be zero, because we focus on the contribution of the pump load only (note: $I_r = Bl_b$)

$$-M_p = \frac{d^2\theta}{dt^2} Bl_b \quad (10.26)$$
Hence, the shearing force on the blade at the hub due to the pump rod force is found by substituting (10.26) in (10.25):

\[ F_{bp} = - \frac{M_p J_b}{I_b} \quad 0 < \theta + \theta_b < \pi \quad (10.27) \]

\[ F_{bp} = 0 \quad \pi < \theta + \theta_b < 2\pi \]

with \( M_p = F_p \frac{1}{2} \sin (\theta + \theta_b) \)

\[ F_{bp} = - F_p \frac{1}{2} \sin (\theta + \theta_b) \frac{J_b}{R I_b} \quad 0 < \theta + \theta_b < \pi \quad (10.28) \]

\[ F_{bp} = 0 \quad \pi < \theta + \theta_b < 2\pi \]

with (10.23) the relation to the rotor torque \( Q_r \) is made and with (10.2) we can rewrite (10.28) into:

\[ F_{bp} = - C_Q \frac{1}{2} \rho v^2 \pi^2 R^3 \sin(\theta + \theta_b) \frac{J_b}{R I_b} \quad 0 < \theta + \theta_b < \pi \quad (10.29) \]

\[ F_{bp} = 0 \quad \pi < \theta + \theta_b < 2\pi \]

In the table of fig. 10.4, the forces and moments calculated above are listed, decomposed in radial, axial and tangential directions.
<table>
<thead>
<tr>
<th>FORCES</th>
<th>MOMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>AXIALLY</strong></td>
<td><strong>TANGENTIALLY</strong></td>
</tr>
<tr>
<td>TORQUE</td>
<td>( \frac{C}{Q} \frac{1}{\rho} \nu^2 \pi R^3 j_b / B I_b )</td>
</tr>
<tr>
<td>WEIGHT</td>
<td>( - mg \sin \theta )</td>
</tr>
<tr>
<td>THRUST</td>
<td>( C_t \frac{1}{\rho} \nu^2 \pi R^2 / B )</td>
</tr>
<tr>
<td>INERTIA</td>
<td>( 2 \cos \theta \Omega_\Omega J_b ) ( \sin \theta \cos \theta \Omega_z^2 J_b ) ( x_z b )</td>
</tr>
<tr>
<td>PUMP LOAD</td>
<td>( -C_Q \frac{1}{\rho} \nu^2 \pi^2 R^3 \sin (\theta + \theta_b) / B I_b J_b ) (1)</td>
</tr>
</tbody>
</table>

\( (1) \): \( 0 < \theta + \theta_c < \pi \)

\( (2) \): \( \pi < \theta + \theta < 2\pi \)

Fig. 10.4 The forces and moments on a rotor blade.
10.3 **Stresses in the rotor spoke at the hub**

Due to the loads calculated in 10.2, bending, shearing and tensile stresses occur at the hub of a rotor blade.
We assume that the spoke has a cross section $A_s$ and a moment of inertia $I_s$.

**SHEARING STRESS**

Axial and tangential forces act in a plane perpendicular to the rotor spokes.
Since axial and tangential forces are perpendicular, the resulting shearing stress $\tau$ can be calculated:

$$\tau = \frac{\sqrt{\left(\sum F_a\right)^2 + \left(\sum F_t\right)^2}}{A_s}$$

(10.30)

This shearing stress $\tau$ is a function of $\theta$ and is assumed to be constant over the cross section in question.

**TENSILE STRESS**

Radial forces on a rotor blade cause a tensile stress $\sigma_t$ which can be calculated from:

$$\sigma_t = \frac{\sum F_r}{A_s}$$

(10.31)

This tensile stress $\sigma_t$ is a function of $\theta$ and is assumed to have a constant value over the cross section in question.
BENDING STRESS

Axial and tangential moments cause bending stress $\sigma_b$ in the rotor blade at the hub.

Axial and tangential moments are perpendicular in a plane perpendicular to the rotor blade axis, hence the resultant bending moment $M_{\text{tot}}$ can be calculated:

$$M_{\text{tot}} = \sqrt{\left(\Sigma M_a\right)^2 + \left(\Sigma M_t\right)^2}$$  \hspace{1cm} (10.32)

The resultant bending moment $M_{\text{tot}}$ causes bending stress $\sigma_b$ in the rotor spokes at the hub:

$$\sigma_b = \frac{M_{\text{tot}}}{I_s} c$$  \hspace{1cm} (10.33)

where $c$ is the distance from the neutral axis of the rotor spoke to the outer fibre.

This bending stress $\sigma_b$ is a function of $\theta$. For arbitrary cross sections, the moment of inertia $I_s$ is directional and $\sigma_b$ depends on the direction of $M_{\text{tot}}$.

For the calculation of rotors, as designed by SWD, having pipes as rotor spokes, $I$ is not directional and the bending stress $\sigma_b$ only depends on the maximum value of $M_{\text{tot}}$ and not on the direction of $M_{\text{tot}}$. 
10.4 Calculation of the combined stresses

The combined effect of shearing stress $\tau$, tensile stress $\sigma_t$ and bending stress $\sigma_b$ can be judged by the so-called Huber-Hencky reference tension $\sigma_{HH}$ which is defined as:

$$\sigma_{HH} = \sqrt{\left(\sigma_b + \sigma_t\right)^2 + 3\tau^2}$$  \hspace{1cm} (10.34)

This reference tension $\sigma_c$ should be equal to or less than the admissible tensile stress of the material.

Substituting (10), (11) and (12) into (13)

$$\sigma_{HH} = \sqrt{\left(\frac{\Sigma M_a}{I_s} + \frac{\Sigma F_a}{A_s}\right)^2 + \frac{\Sigma F_t^2}{A_s^2} + 3 \left(\frac{\Sigma F_r^2}{A_s^2}\right)}$$  \hspace{1cm} (10.35)

Substituting for $\Sigma M_a$, $\Sigma M_r$, $\Sigma F_a$, $\Sigma F_t$ and $\Sigma F_r$ the expressions derived in section 10.2, $\Sigma_{HH}$ can be calculated as a function of $\theta$ and $\theta_c$.

In practice it appears that for slow-running windmills, with tip speed ratios up to 2, such as designed by SWD, the equation for $\sigma_{HH}$ can be simplified by realising that stresses in the rotor blades at the hub are largely determined by moments acting on the blades and that the effect of the forces acting on the blades may be neglected.

This is shown in appendix B, where for a representative rotor, the WEU I-3, a six-bladed 3 m. diameter slow running ($\lambda = 2$) rotor, the forces and moments have been calculated.

Neglecting the forces, equation (10.35) can be rewritten into:

$$\sigma_{HH} = \sqrt{\left(\frac{\Sigma M_a}{I_s}\right)^2 + \left(\frac{\Sigma M_r}{A_s}\right)^2} \cdot \frac{\Sigma}{c}$$  \hspace{1cm} (10.36)
or, when substituting the expressions of section 10.2:

\[
\sigma_{HH} = \sqrt{\left[ \frac{mgL \sin \theta + C_Q \beta \rho v^2 \pi^2 R^3}{B} \right] + \left[ \frac{1}{3} C_t \rho v^2 \pi R^3/B + 2 \cos \theta \Omega_x \Omega_z I_r \right]^2 \frac{I_B}{c} \right]}
\]

\[
\sigma_{HH} = \sqrt{\left[ mgL \sin \theta \right]^2 + \left[ \frac{1}{3} C_t \rho v^2 \pi R^3/B + 2 \cos \theta \Omega_x \Omega_z I_r \right]^2}
\]

\[
\pi < \theta + \theta_b < 2\pi
\]

From eq. (13) it follows that the maximum of \( \sigma_{HH} \) coincides with the maximum of the resultant moment \( M_{tot} \):

\[
M_{tot} = \sqrt{(\Sigma M_a)^2 + (\Sigma M_r)^2}
\]

The direction of \( M_{tot} \), denoted by the angle \( \psi \) between \( M_{tot} \) and the rotation axis of the rotor, is:

\[
\psi = \arctg \frac{\Sigma M_r}{\Sigma M_a}
\]

The maximum of \( M_{tot} \) is a function of the position of the blade, denoted by the angle \( \theta \) and of the position of the blade relative to the crank.

In fig. 10.5 to fig. 10.8 the magnitude and direction of \( M_{tot} \) is illustrated for the WEU I-3 rotor.
The calculations are based on the following data:

- Rotor diameter \( D = 3 \text{ m} \)
- Mass of one blade \( m = 4.6 \text{ kg} \)
- Max. angular velocity of rotor \( \Omega_x = 15.7 \text{ rad/sec.} \)
- Max. angular velocity of head \( \Omega_z = .63 \text{ rad/sec.} \)
- Mass moment of inertia \( I = 3.6 \text{ kgm}^2 \)
- First mass moment of inertia \( J = 3.45 \text{ kgm} \)
- Undisturbed wind speed \( V = 12 \text{ m/s} \)
- Torque coefficient \( C_Q = .19 \)
- Thrust coefficient \( C_t = 8/9 \)
- Rotor spoke (OD * ID) \( \phi 27 \ast \phi 22 \text{ mm} \)

**FORCES:**

\( a. \text{ Axially} \)

- Thrust \( F_{bt} = C_t \frac{1}{2} p V^2 \pi R^2/B = 90.4 \text{ N} \)
- Gyro effect: \( F_{bia} = 2 \cos \theta \Omega_x \Omega_z J = 68 \cos \theta \text{ N} \)

\( b. \text{ Tangentially} \)

- Torque: \( F_{bq} = \frac{C_Q \frac{1}{2} p V^2 \pi R^3 J_b}{B I_b} = 27.8 \text{ N} \)
- Weight: \( F_{bg} = - mg \sin \theta = - 46 \sin \theta \text{ N} \)
- Gyro effect: \( F_{bit} = \Omega^2 J_b \sin \theta \cos \theta = 1.4 \sin \theta \cos \theta \text{ N} \)
- Pump load: \( F_{tp} = - \frac{C_Q \frac{1}{2} p V^2z^2 R^3 \sin(\theta + \alpha)I_b}{B I_b} = - 87 \sin(\theta + \alpha) \text{ N} \)

\( 0 < \theta + \alpha_b < \pi \)

\( F_{tp} = 0 \quad \pi < \theta + \alpha_b < 2\pi \)
c. Radially

Weight: \[ F_{bgr} = mg \cos \theta = 46 \cos \theta \text{ N} \]

Gyro effect: \[ F_{bir} = \Omega^2 J_x \sin^2 \theta \text{ N} \]

\[ I_b = 850 + 1.4 \sin^2 \theta \text{ Nm} \]

MOMENTS

a. Axially

Weight: \[ M_{bga} = - mgL \sin \theta = - 34.5 \sin \theta \text{ Nm} \]

Gyro effect: \[ M_{bia} = \sin \theta \cos \theta \Omega^2 I_b = 1.4 \sin \theta \cos \theta \text{ Nm} \]

Pump load: \[ M_{bp} = - C_Q \rho \frac{V^2 \pi R^3}{B} \frac{\sin(\theta + \theta_b)}{\theta} = 91 \sin(\theta + \theta_b) \]

\[ 0 < \theta + \theta_b < \pi \]

\[ M_{bp} = 0 \]

\[ \pi < \theta + \theta_b < 2\pi \]

b. Tangentially

Thrust: \[ M_{bt} = \frac{1}{3} C_t \rho \frac{V^2 \pi R^3}{B} = 90 \text{ Nm} \]

Gyro effect: \[ M_{bit} = 2 I_b \cos \theta \Omega \frac{\Omega}{x \ z} = 71 \cos \theta \text{ Nm} \]

From the numerical values of forces and moments as calculated above, we may conclude:
1. The resulting axial and tangential shearing force is equal or less than 300 N.
   For the rotor spoke with a cross sectional area of 192 mm$^2$ this means a shearing stress of less than 2 N/mm$^2$. This can be neglected with respect to the maximum admissible shearing stress of mild steel.

2. The effect of forces on the rotor can be neglected with respect to the effect of moments.
   Only the gyroscopical force $\Omega^2 x I$ contributes significantly to the stress level, but still this effect can be neglected with respect to the stresses due to the imposed moments.

3. Determining the resultant bending moment $M_{tot}$ from fig. 10.7 to be 187 Nm, the Huber Hencky reference tension can be calculated for the WEU I-3 rotor.

   This reference tension is:
   
   $$\sigma_{HH} = \frac{M_{tot} \times c}{I} = 173 \text{ N/mm}^2$$

   Considering:
   - a high level of bending stress
   - stress raisers such as welds, notches and possible corrosion
   - the cyclic character of the load
   we must conclude that the scantlings of the rotor spoke are too small.

?? Fig. 10.6 shows the magnitude of the resultant bending moment at the design wind speed 4 m/s and at 7 m/s, a wind speed at which the rotor is assumed to start turning out of the wind.
   The blade in question has an angle $\theta_c = 0$ with the crank, and there is no yawing of the head.
Fig. 10.5 The resultant bending moment $M_{\text{tot}}$ as a function of the position of the blade in the rotor plane for wind speeds of 4 and 7 m/s.

The direction $\psi$ of $M_{\text{tot}}$ at the same conditions as in fig. 10.4 is illustrated in fig. 10.6.

Fig. 10.6 The direction $\psi$ of the resultant bending moment $M_{\text{tot}}$ as a function of the position of the blade in the rotor plane for wind speeds of 4 and 7 m/s.
The ultimate load conditions are illustrated in fig. 10.7 and 10.8. The ultimate load conditions are assumed to occur when a 12 m/s gust hits the rotor which simultaneously yaws out of the wind with 0.1 revolution per second (0.63 rad/s).

Fig. 10.7 shows $M_{tot}$ for a blade at an angle $\theta_c = 0$ with the crank, for clockwise and anti-clockwise yawing of the head.

![Graph showing resultant bending moment $M_{TOT}$ as a function of position of the blade in the rotor plane at the ultimate load conditions.]

Fig. 10.7 The resultant bending moment $M_{tot}$ as a function of the position of the blade in the rotor plane at the ultimate load conditions.
From equation (10.39) it follows that the maximum of $M_{tot}$ occurs when thrust and gyroscopic moment have the same direction (sign). Since thrust is constant (for a specific wind speed), and positive, the ultimate load occurs when the gyroscopic moment is positive.

This can be taken into account by using the absolute value of the gyroscopic moment in eq. (16).

This is illustrated in fig. 10.8, where $M_{tot}$ is illustrated as a function of $\theta$ for the eight blades of the WEU I-3.

The discontinuity at $\theta = 90^\circ$ and $270^\circ$ is due to using the absolute value of the gyroscopic moment which can be realised when comparing figs. 10.7 and 10.8.

Fig. 10.8. Resultant bending moment $M_{tot}$ for the six blades of the WEU I-3 rotor as a function of the position of the blades in the rotor plane at ultimate load conditions.

As can be seen in fig. 10.7, the maximum bending moment in the blades varies for different blades and the blade with an angle $\theta_c = 60^\circ$ experiences the highest bending moment.
CONCLUSIONS

Even in the case of an assumed constant wind speed over the rotor area, the loads on a rotor blade are cyclic and gradually a sufficiently large number of repetitions will be built up which may lead to a fatigue break.
Hence when determining the scantlings of a rotor spoke, stresses should be kept below the admissible fatigue stress, which can be found using Wöhler curves [29].

Stresses must be calculated by means of the formula of Huber-Henckey, corrected for stress concentrations, surface irregularities and corrosion. Due to such stress raisers, the maximum stress in a cross section may be several times the calculated maximum stress, and emphasis must be placed upon avoiding such stress raisers by careful design, and surface treatments such as painting or galvanising.
11. SAFETY SYSTEMS

Windmills without a safety system usually have a short life. The history of wind energy, even the recent history, is scattered with tragic incidents where wind machines have been blown into pieces because their safety system was not present at all or badly designed.

In this chapter we shall first discuss the various possibilities to protect a windmill against too high wind speeds and afterwards analyze the hinged vane safety system, widely used on water pumping windmills.

It must be mentioned that there are exceptional cases of some very small (D < 1 m) wind turbines which are constructed so sturdy that they can withstand speeds up to 40 m/s or more without any safety system. This is the case with small battery chargers used on sailing vessels. The resulting very high costs per kW indicate that this method is unacceptable for any wind machine larger than a metre in diametre.

11.1 Survey of different safety systems

Each safety system must perform two functions:

1. Limit the axial thrust forces on the rotor.

The reason is clear: at high wind speeds the bending moment on the blades becomes too high and eventually they will break. In the case of windmills with weak towers (or with guy wires as is the case with vertical axis windmills) the tower might fail even before the rotor blades actually come down.
2. Limit the **rotational speed** of the rotor.

High rotational speeds lead to the following phenomena:

- High centrifugal forces, resulting in high tensile forces in the blades. Finally one of the blades will be launched as a projectile, leaving behind an unbalanced wind machine with an extremely short lifetime.

- A combination of high rotor speed and sudden directional changes of the rotor head gives rise to high gyroscopic moments, i.e. high bending moments in the blades and the rotor shaft.

- High tip speeds can induce a dangerous aero-elastic behaviour, called "flutter". This is a combination of severe torsional and bending vibrations in the blades.

- In the case of water pumping windmills the high pump frequencies lead to a sharp increase of the shock forces, carried to the bearings and the crank mechanism via the pump rod. They are caused by acceleration forces and extra shock forces due to delayed closure of the valves of the piston pump.

The safety systems can act either on the rotor as a whole or on each of the blades. The first method is usually employed with the multiblade rotors such as those on water pumping windmills, and the second method is mostly used for fast running two- or three-bladed wind turbines. The different designs in each group are listed below.
1. Turning the rotor sideways
   1.1 Inclined hinged vane
       (eccentric rotor or auxiliary vane balances vane)
   1.2 Ecliptic control
       (eccentric rotor balances spring-loaded vane)
   1.3 Pressure plate
       (plate dislocks the vane)
2. Turning the rotor upward
   (eccentric rotor balances a weight)
3. Brake flaps, separate from the blades
   (brake action and spoiling rotor flow)

1. Pitch control
   (changing setting angle of blades, positive or negative)
   1.1 Centrifugal weights
   1.2 Axial forces on the blade
   1.3 Externally operated (hydraulic or servo)
2. Stall of blades
   (for constant speed turbines with fixed pitch blades)
3. Brake flaps at the tip of the blades
   3.1 Flap axis parallel to blade axis
   3.2 Flap axis perpendicular to blade axis
4. Spoilers
   (movable ridges that spoil part of the blade's performance)

It will be clear that this list is by no means exhaustive. Its main purpose is to give a rough idea about the wide range of possibilities in the design of safety systems.
11.2  Hinged vane safety system

11.2.1 General description

The first description of the inclined hinged vane safety system, widely used on slow running water pumping windmills, has been given by Kragten [30]. It is basically a static description, yielding the angle of yaw of the rotor as a function of the undisturbed wind speed. A dynamical model of this safety system, which turns out to be rather difficult, is currently being developed at the University of Amsterdam [31]. In this section we shall base ourselves upon the simpler static model.

Basically, the wind rotor can be pushed out of the wind by two methods (apart from the side force on the rotor itself): with an auxiliary vane attached to the head of the windmill or by placing the rotor eccentric with respect to the vertical rotation axis of the head. In order to analyse both methods they are both included in the model and shown in fig. 11.1. A practical windmill will have either an auxiliary vane or an eccentric rotor. To distinguish the vane from the auxiliary vane, we will use the indication "main vane".

The function of the safety system with an inclined hinged vane is to limit the rotational speed of the rotor and to limit the axial forces acting upon the rotor. This is accomplished by turning the rotor gradually out of the wind with increasing wind speed. As the vane remains more or less parallel to the wind, this turning of the head implicates that the vane is turned around its inclined hinge, thereby being lifted. The vane strives towards its lowest position, however, providing the moment that balances the moments of rotor and auxiliary vane.

In the static analysis presented here the position of rotor and vane are stable at every wind speed, i.e. the moments around the hinge axis and around the vertical axis of the rotor head do balance:
hinge axis: the aerodynamic forces on the main vane plus the weight force together yield an oblique downward force. The main vane will move under the influence of this force until the force points in the direction of the hinge axis. At that point, the moment, due to the aerodynamic forces, is balanced by the moment due to the weight of the vane.

vertical axis: the aerodynamic forces on the main vane exert a moment around the vertical axis which is balanced by the moment of the aerodynamic forces on the rotor and the auxiliary vane.

With increasing wind speed the aerodynamic forces on the rotor and auxiliary vane increase, turning the rotor further out of the wind and forcing the main vane further from its lowest position (fig. 11.2).
Fig. 11.1 Schematic drawings of the different parameters involved in the analysis of the inclined hinged vane safety system.
Fig. 11.2 The behaviour of the inclined hinged safety system with increasing wind speed. The yawed position at $V = 0$ shows that the hinge axis in this case possesses a preset angle $\delta_0$. This preset angle of yaw causes the rotor to face the wind perpendicularly at the design speed $V_d$. The windmill is seen from above.
11.2.2 Forces

11.2.2.1 Forces on the rotor

If the angle of yaw between the rotor axis and the wind direction is \( \delta \), then the wind speed \( V \) can be seen as the vectorial sum of a wind speed \( V \cos \delta \) perpendicular to the rotor plane and a wind speed \( V \sin \delta \) parallel to the rotor plane.

\[ V \]

\[ V \cos \delta \]

\[ V \sin \delta \]

Figure 11.3 The forces on a rotor in yaw, with yawing angle \( \delta \).

The axial force, or thrust, on the rotor, with swept area \( A_r \) can be written as:

\[ F_{rt} = C_t \frac{1}{2} \rho (V \cos \delta)^2 A_r \] (11.1)

The dimensionless thrust coefficient \( C_t \) varies with the tip speed ratio \( \lambda \) of the rotor, but for \( \lambda_0 < \lambda < \lambda_{\text{max}} \) it can be approximated by \( C_t = 8/9 \).

For the side force on the rotor a similar expression can be written:

\[ F_{rs} = C_f \frac{1}{2} \rho (V \sin \delta)^2 A_{rs} \] (11.2)

Here \( A_{rs} \) is the area of the rotor projected sideways. See fig. 11.1. We will assume that \( C_f \) is constant and equal to 8/9, although in practice \( C_f \) is a function of \( \lambda \).
11.2.2.2 Aerodynamic forces on the vane

Any flat plate, such as a vane, experiences lift and drag forces when placed in a flow of air: it behaves as a crude airfoil. As the pressure forces dominate the behaviour of such flat plates the resultant force $F_v$ is nearly always perpendicular to the plate.

\[ \text{(1-a) } \cdot \mathbf{V} \]

Figure 11.4 The aerodynamic force on a square plate is nearly always perpendicular to the surface of the plate.

For flat plates the dimensionless normal force coefficient $C_N$ is given in fig. 11.5.

\[ C_N \]

Fig. 11.5 The dimensionless normal force coefficient $C_N$ of a square plate.
For our purpose the range \(0 < \alpha < 40^\circ\) is important, in which \(C_N\) of a square plate can be approximated by a linear function of \(\alpha\):

\[
C_N = 2.6 \times \alpha \text{ (\(\alpha\) in radians)} \tag{11.3}
\]

As a result the normal force of a wind speed \((1-\alpha)V\), felt by the vane, is given by:

\[
F_v = C_N \times \frac{1}{2} \rho (1-\alpha)^2 \sqrt{V^2 A_v} \tag{11.4}
\]

or

\[
F_v = 2.6 \times \alpha \times \frac{1}{2} \rho (1-\alpha)^2 \sqrt{V^2 A_v}
\]

In the preceding lines we used a value \((1-\alpha)V\) for the wind speed that the main vane experiences. This is because the rotor slows down the wind in order to extract energy from it. A consequence is that the value of \((1-\alpha)\) changes with the load of the rotor or, in other words, with the wind speed. Estimating the value of \((1-\alpha)\) therefore becomes rather difficult.

Some information is contained in the first report on the hinged vane safety system by the University of Amsterdam [31], although the data from the \(\lambda_0 = 2\) rotor do not cover tip speed ratios higher than 2. Extrapolation of the data beyond \(\lambda = 2\) clearly points towards values of \((1-\alpha)\) between 0.6 and 0.8 for angles of attack of the vane \(\alpha = 10^\circ\) to \(15^\circ\). This is shown in fig. 11.6. It is also clear from this figure that \((1-\alpha)\) strongly depends on \(\delta\), the angle of yaw of the rotor. For high values of \(\delta\), i.e. \(\delta > 30^\circ\), the vane clearly "feels" the undisturbed wind speed at low tip speed ratios, this because the wake of the rotor is bent by the rotor, away from the vane. For higher tip speed ratios this effect is less pronounced and it seems that \((1-\alpha) = 0.7\) is a reasonable value for high tip speed ratios, irrespective of the angle of yaw.
Fig. 11.6 The factor $(1-a)$, indicating the reduced wind speed behind the rotor, as a function of the tip speed ratio of a rotor with a design tip speed ratio of 2. The value of $(1-a)$ is found by taking the square root of the ratio between the moment on the vane with rotor and without rotor. Deflection of the wake, however, causes unrealistic values for high values of $\delta$ at low $\lambda$. 
11.2.2.3 Forces due to the weight of the vane

The weight of the vane plus the vane arm gives a vertical downward force $G$, acting upon the centre of gravity of the vane plus vane arm. This centre of gravity is located at a distance $R_G$ from the hinge axis of the vane.

The (hinge) $s$-axis and the (vertical) $z$-axis determine a vertical plane $DEFG$ (fig. 11.7). The force $G$ lies in a plane $D'E'F'G'$ parallel to this vertical plane. The centre of gravity of the vane moves in the plane $EE'D'D$, so the component of the force $G$ in the plane $D'E'F'G'$ is $G \sin \varepsilon$ (fig. 11.7).

![Diagram](image)

Fig. 11.7 Vector diagram of the force on the main vane due to the weight $G$ of the vane and vane arm.

From this force $G \sin \varepsilon$ only a fraction $G \sin \varepsilon \sin \gamma$ acts in the direction perpendicular to the vane if the angle between the vane and its lowest position is $\gamma$ (in plane $EE'D'D$); see also fig. 11.8b). We conclude that the force perpendicular to the main vane due to its weight can be written as:

$$F_{vg} = G \sin \varepsilon \sin \gamma \quad (11.5)$$

This force acts on a distance $R_G$ from the (hinge) $s$-axis.
11.2.2.4 Forces on the auxiliary vane

The auxiliary vane is often located in the plane of the rotor at a distance of about 1.5 R from the rotor shaft. This implies that the auxiliary vane normally "feels" the undisturbed wind speed, except for large angles of yaw when the auxiliary vane comes in the wake of the rotor. Here we assume that the auxiliary vane always experiences the undisturbed wind speed. In 11.3 we shall see that a correction is necessary.

The position of the auxiliary vane is not necessarily parallel to the rotor plane, but generally is at an angle $\xi$ with the rotor plane. The angle $\xi$ is considered positive when the vane is bent backwards i.e. towards the main vane. The result is that the angle of attack of the auxiliary vane is $90^\circ - \delta - \xi$ at an angle of yaw $\delta$ of the rotor. The force on the auxiliary vane, perpendicular to the vane arm, is given by (the value of $C_N$ is given as $C_N(90-\delta-\xi)$):

$$F_a = C_N(90-\delta-\xi) \frac{1}{2} \rho V^2 A_a \cos \xi$$  (11.6)

The normal force coefficient is given in fig. 11.5. One sees that $C_N$ of a square plate remains more or less constant for:

$$90^\circ - \delta - \xi < 35^\circ \quad \text{or} \quad \delta > 55 - \xi$$

With negative values of $\xi$ (an auxiliary vane bent towards you when looking at the windmill with the wind in your back) this condition means that for most angles $\delta$ the force on the auxiliary vane remains constant.

With positive values of $\xi$, the angle of yaw $\delta$ quite easily reaches values such that $C_N$ jumps to its peak value (see fig. 11.5). The result is that in gusty winds a windmill with such a vane turns faster out of the wind but also returns faster at shifts of wind direction. An advantage is also that at very high wind speeds and $\delta = 90^\circ$, the auxiliary vane develops a lift force that pushes the rotor back into the wind just enough to keep it turning slowly. This facilitates a more continuous rotation of the rotor, as is observed at the field test stand in Eindhoven with $\xi = 25^\circ$. 

11.2.2.5 Summary of the forces

All forces mentioned are repeated below for convenience of the reader:

aerodynamic thrust on the rotor:

\[ F_{rt} = C_t \frac{1}{2} \rho (V \cos \delta)^2 A_r \]  
(11.1)

aerodynamic side force on the rotor:

\[ F_{rs} = C_f \frac{1}{2} \rho (V \sin \delta)^2 A_{rs} \]  
(11.3)

aerodynamic normal force on vane:

\[ F_v = 2.6 a \frac{1}{2} \rho (1-a)^2 V^2 A_v \]  
(11.4)

weight normal force on vane:

\[ F_{vg} = G \sin \varepsilon \sin \gamma \]  
(11.5)

aerodynamic force on auxiliary vane:

\[ F_a = C_N(90^\circ - \delta - \xi) \frac{1}{2} \rho V^2 A_a \cos \xi \]  
(11.6)
11.2.3 Moments

In a stationary situation an equilibrium exists at any wind speed between the moments exerted by aerodynamic forces on the rotor, the vane and the auxiliary vane and by the force due to the weight of the vane.

The vane will remain more or less parallel to the wind. The increasing thrust on rotor and auxiliary vane attempts to turn the rotor out of the wind, forcing the vane to turn around its hinge axis and thereby lifting the vane.

Basically two equilibria exist, one around the z-axis and one around the s-axis.

**z-axis:**

- moments of aerodynamic forces on rotor and auxiliary vane
- moment of aerodynamic force on vane

**s-axis:**

- moment of aerodynamic force on vane
- moment of weight of vane

In the following we shall work out these two equilibria in detail.

The moments around the hinge axis are determined by the aerodynamic force and the weight force perpendicular to the vane (see fig. 11.8).

\[ F_v \cdot R_v = F_{vg} \cdot R_G \]  \hspace{1cm} (11.7)

\[ 2.6 \alpha \ \frac{1}{2} \rho (1-a)^2 V^2 \ A_v \ R_v = G \sin \epsilon \sin \gamma R_G \]  \hspace{1cm} (11.8)
The distance $R_v$ is the distance from $1/4$ of the width of the front edge of the main vane to the hinge axis, while $R_c$ is the distance from the centre of gravity of the vane and the arm to the hinge axis (see fig. 11.1).

Around the vertical axis the moments caused by the aerodynamic forces on the rotor and the auxiliary vane are balanced by the moment exerted by the vane:

$$M_z(F_{rt}) + M_z(F_{rs}) + M_z(F_a) = M_z(F_v) \quad (11.9)$$

The three terms at the left-hand side of the equation can be found rather easily:

$$M_z(F_{rt}) = C_t \frac{1}{2} \rho (V \cos \delta)^2 A_r e \quad (11.10)$$

$$M_z(F_{rs}) = C_f \frac{1}{2} \rho (V \sin \delta)^2 A_{rs} f \quad (11.11)$$

$$M_z(F_a) = C_N(90^\circ - \delta - \xi) \cos \xi \frac{1}{2} \rho V^2 A_a R_a \quad (11.12)$$

The right-hand term can be found by selecting the components of $F_v$ that act in a direction perpendicular to the $z$-axis (fig. 11.9). The first component is $F_v \cos \gamma$ acting on an arm $(h + R_v \cos \gamma \cos \varepsilon)$ and the second component is $F_v \sin \gamma \cos \varepsilon$ acting on an arm $R_v \sin \gamma$.

The result is:

$$M_z(F_v) = F_v \cos \gamma (h + R_v \cos \gamma \cos \varepsilon) + F_v \sin \gamma \cos \varepsilon (R_v \sin \gamma)$$

$$M_z(F_v) = F_v (h \cos \gamma + R_v \cos \xi) \quad (11.13)$$
Fig. 11.8 The force components due to the weight of the main vane.
Fig. 11.9  The force components of $F_v$, due to the aerodynamic forces on the main vane.
11.2.4 Solving the moment equations

A combination of the equations given in the preceding sections yields the two moment equations, one for the moments around the vertical axis.

\[
C_t \frac{1}{2} \rho (V \cos \delta)^2 A_r e + C_f \frac{1}{2} \rho (V \sin \delta)^2 A_{rs} f + C_N(90^\circ - \delta - \xi) \cos \xi \frac{1}{2} \rho V^2 A_v R_v = 2.6 \alpha \frac{1}{2} \rho (1-\alpha)^2 V^2 A_v (h \cos \gamma + R_v \cos \varepsilon)
\]  

and one for the moments around the hinge axis:

\[
2.6 \alpha \frac{1}{2} \rho (1-\alpha)^2 V^2 A_v R_v = G \sin \varepsilon \sin \gamma R_G
\]  

The solution of these equations should be of the form:

\[
\delta = \delta(V)
\]

In order to see if this is possible, we shall first give a survey of the parameters that should be known beforehand:

**Windmill parameters:** lengths: \( R_v \ R_a \ R_e \ R_f \ R_h \)

areas: \( A_r \ A_{rs} \ A_a \ A_v \)

weight: \( G \)

angles: \( \varepsilon \ \xi \)

**Aerodynamic parameters:** coefficients: \( C_t \ C_f \ C_N \)

factor: \( (1-\alpha) \)

constant: \( \rho \)
This leaves us with the unknown parameters:

angles: \( \alpha, \gamma, \delta \)

wind speed: \( V \)

With four parameters left and two equations we still have one parameter too much in order to find a relation \( \delta = \delta(V) \). The angle \( \alpha \), however, between the vane and the wind speed can be expressed in \( \delta, \epsilon \) and \( \gamma \) and this will enable us to solve the problem.

The wind speed \((1-a)V\) at the vane can be resolved into two wind speeds in a horizontal plane: \((1-a)V \sin \delta\) and \((1-a)V \cos \delta\) (fig. 11.10). The components of these speeds in the plane of the vane movement are \((1-a)V \sin \delta\) and \((1-a)V \cos \delta \cos \epsilon\). The components of these velocities perpendicular to the vane are: \((1-a)V \sin \delta \cos \gamma\) and \((1-a)V \cos \delta \cos \epsilon \sin \gamma\), pointing in opposite directions. The sum of these velocities must be equal to \((1-a)V \sin \alpha\), giving us the relation we were looking for:

\[
\sin \alpha = \sin \delta \cos \gamma - \cos \delta \cos \epsilon \sin \gamma
\]

(11.17)

Fig. 11.10 Illustration for the procedure to find an expression for the angle \( \alpha \) between the vane and the wind speed \((1-a)V\).
Now we can eliminate $\alpha$ from the moment equations (11.14) and (11.15) realizing that:

$$\alpha = \sin \alpha \text{ for small angles } \alpha \text{ (in radians)} \quad (11.18)$$

The accuracy of this approximation is better than $2\%$ for $\alpha < 18^\circ$.

With (11.17) the second moment equation transforms into:

$$2.6(\sin \delta \cos \gamma - \cos \delta \cos \epsilon \sin \gamma) \frac{1}{2} p (1-a)^2 V^2 A_v R_v =$$

$$G \sin \epsilon \sin \gamma R_G$$ \quad (11.19)

We can rewrite this equation as an expression of $\gamma$, necessary to eliminate $\gamma$ from both moment equations.

$$\frac{\sin \gamma}{\sin \delta \cos \gamma - \cos \delta \cos \epsilon \sin \gamma} = \frac{2.6 \frac{1}{2} p (1-a)^2 V^2 A_v R_v}{G R_G \sin \epsilon}$$ \quad (11.20)

Introducing the function $F$ with:

$$F = \frac{G R_G \sin \epsilon}{2.6 \frac{1}{2} p (1-a)^2 V^2 A_v R_v}$$ \quad (11.21)

We can rewrite (11.20) as follows:

$$\frac{\sin \delta}{\tan \gamma} - \cos \delta \cos \epsilon = F$$ \quad (11.22)

or

$$\tan \gamma = \frac{\sin \delta}{F + \cos \delta \cos \epsilon} = T$$ \quad (11.23)

From $\tan \gamma$ we find the values of $\sin \gamma$ and $\cos \gamma$ with:

$$\sin \gamma = \frac{T}{\sqrt{1 + T^2}} \quad \text{and} \quad \cos \gamma = \frac{1}{\sqrt{1 + T^2}}$$
Substituting these expressions in the first moment equation yields (the left-hand side remains identical):

left-hand side = \(2.6 \frac{F}{\sqrt{1 + T^2}} \frac{h}{\sqrt{1 + T^2}} + \frac{R_v \cos \epsilon}{\sqrt{1 + T^2}}\)  

\((11.24)\)

We can transform this equation into a dimensionless equation by dividing by \(\frac{h}{\sqrt{1 + T^2}} R_v\) and the complete equation becomes:

\[2.6 \frac{F}{\sqrt{1 + T^2}} (1-a)^2 \frac{h}{\sqrt{1 + T^2}} + \cos \epsilon\]  

\((11.25)\)

With \(F\) and \(T\) (which are both functions of \(V\)) given by \((11.21)\) and \((11.23)\), respectively. We see that it will be a difficult job to transform this into an expression of the form \(\delta = \delta(V)\) so in practice a value of \(V\) (or \(\delta\)) will be given and find the corresponding value of \(\delta\) (or \(V\)) by trial and error, or with a simple calculator program.

Note that, in case the rotor experiences a preset angle \(\delta_0\) the forces on the rotor and the auxiliary vane have to be calculated with \((\delta - \delta_0)\) instead of \(\delta\), affecting only the left-hand side of \((11.25)\). Also the side force becomes negative for \((\delta - \delta_0) < 0\), i.e. \(\sin^2 \delta\) must be replaced by \(\sin (\delta - \delta_0)\) or \(\sin (\delta - \delta_0)\).

In a field situation it will be difficult to measure \(R_v\), \(R_v\) and \(h\), because they require the exact location of the imaginary extension of the hinge axis. The distances can be transformed, however, into distances that are more easy to measure, as indicated in fig. 11.11.
Figure 11.11 The values of $h$, $R_v$, and $R_G$ can also be found by measuring $i$, $r_v$ and $r_G$.

It may be concluded that:

$$R_v = r_v \cos \epsilon$$  \hspace{1cm} (11.26)

$$R_G = r_G \cos \epsilon$$  \hspace{1cm} (11.27)

The value of $h$ is found via the triangle PIJ:

$$h = i + PN$$

$$\sin \epsilon = \frac{PN}{PI}$$

$$\tan \epsilon = \frac{PI}{R_v}$$

$$PN = R_v \sin \epsilon \tan \epsilon$$

$$h = i + r_v \sin^2 \epsilon$$  \hspace{1cm} (11.28)
With these new distances the general equation (11.25) changes into:

\[
C \frac{\cos^2 \epsilon}{T \cos \epsilon A_r^e} v + C \frac{\sin^2 \delta}{\cos \epsilon A_r f} v + C_n (90^\circ - \delta - \xi)^* \\
\cos \frac{\epsilon}{\cos \epsilon A_v} = \frac{2.6 F}{(1+T^2)} \left(1-a\right)^2 \frac{1 + r_v \sin^2 \epsilon}{r_v \cos \epsilon \sqrt{1+T^2}} + \cos \epsilon
\]

The expression for F becomes:

\[
F = \frac{G r_g \sin \epsilon}{2.4 \frac{1}{2} \rho \left(1-a\right)^2 v^2 A_r v}
\]

11.2.5 Theory versus practice

The expressions for the behaviour of the hinged vane safety system, as derived above, give only a crude description of the reality. The forces on the main vane and the auxiliary vane are affected by the changes in direction and speed in the wake of the rotor and not only by a constant factor \((1-a)^2\). The thrust and side forces on the rotor depend upon the tip speed ratio of the rotor in a manner which is not clearly known yet.

For these reasons it is instructive to compare the wind tunnel results of a scale model with the theoretical results, calculated with the above formula for the same scale model. The wind tunnel tests were carried out by Bos, Schoonhoven and Verhaar (University of Amsterdam) in 1981 and the parameters of their model were as follows:
Parameters of model of SWD 2740 windmill

\[ R = 0.75 \text{ m} \quad A_r = 1.77 \text{ m}^2 \quad \delta_o = 0^\circ \]

\[ R_v = 1.00 \text{ m} \quad A_v = 0.41 \text{ m}^2 \quad \varepsilon = 15^\circ \]

\[ R_a = 1.26 \text{ m} \quad A_a = 0.07 \text{ m}^2 \quad \xi = 0^\circ \text{ or } +25^\circ \]

\[ R_G = 0.80 \text{ m} \quad G = 32.4 \text{ N} \]

\[ h = 0.14 \text{ m} \]

\[ e = 0 \text{ m} \]

Dimensions vane : height 0.55 m chord 0.75 m (h/c = 0.75)

Dimensions aux. vane : height 0.19 m chord 0.38 m (h/c = 0.50)

In the calculation, the characteristics of the auxiliary vane (see fig. 11.5) are approximated by a constant value of \( C_N = 1.6 \) for angles of attack from 45° up to 90° and a linear increase of \( C_N \) from the origin up to a value of \( C_N = 2.0 \) at 45°. The following constants were used: \((1-a) = 0.7\), \( \rho = 1.2 \text{ kg/m}^3 \), \( C_t = 8/9 \) and \( C_f = 8/9 \).

The results of measurements and calculations are shown in fig. 11.12. Both graphs clearly show that the model is only correct for low wind speeds up to about 5 m/s. For high wind speeds the model grossly deviates from reality: much higher wind speeds are necessary to turn the rotor over a given angle \( \delta \) than the model predicts. One of the reasons is that the auxiliary vane of the model is mounted quite close to the rotor. The result is that at low values of \( \delta \), the auxiliary vane is already influenced by the wake of the rotor, causing a reduction in speed and change in direction of the wind speed. In a first (very rough) approximation the resulting reduction of the forces on the auxiliary vane can be
respected by multiplication with a factor \( \cos \delta \). The result is shown in fig. 11.12 as well. The improvement for the \( \xi = 0^\circ \) case is considerable, but less impressive for the \( \xi = 25^\circ \) case. The discontinuity in the theoretical curves stems from the discontinuity in the \( C_N - \alpha \) graph of the auxiliary vane.

These examples are given to stress that the theory presented here is by no means complete, but that it may serve to give more insight in the behaviour of the hinged vane safety system.
Figure 11.12 Theoretical and measured results of a scale model of the SWD 2740 rotor. The wind tunnel data were taken by Bos, Schoonhoven and Verhaar of the University of Amsterdam (1981). In the model (1-a) is taken to be 0.7. The correction indicated consists of multiplying the forces on the auxiliary vane with \( \cos \delta \).
12. COSTS AND BENEFITS

12.1 General

The primary goal of any financial analysis is to find out whether the benefits of an activity outweigh its costs. In the case of wind energy systems the costs are relatively easy to determine once basic assumptions about money costs, escalation rates, etc. are made, but the benefits are not always easy to calculate. The agricultural benefits of pumped water depend on the type of crops, the availability of water in critical months, the seasonal variations in the available water quantities, etc. We will assume here that it is irrelevant whether the water is pumped by a diesel pump set or a windmill and regard the (not spent) costs of pumping water by an alternative system as the benefits for the windmill. That means, we assume that the energy has to be produced anyway and that we have to calculate the maximum allowable investment for a windmill to produce the same amount of energy.

Note that this approach does not hold in cases where the unpredictability of the wind power has urged the farmer to plant relatively more drought-resistant crops (with less profit) or even no crops at all in risky months. Apart from this the decision whether or not to invest in a windmill can depend on many factors other than a cost/benefit analysis, such as employment, saving foreign exchange, uncertainties in fuel supply, agricultural support programmes, etc.

In our calculations we will mainly use the so-called "present value" method to compare cost with benefits. It is a proper method to decide whether windmills are a sound investment within the basic assumptions about interest rates etc. The resulting cost figures, however, include the effect of interest and inflation throughout the lifetime of the windmill (or diesel), transferred to the present. They are not the actual costs of the equipment after one year, after two years etc. Because potential buyers tend to compare investments on the basis of first-year costs, we will mention these costs as well, noting that they usually favour the low-investment fuel consuming systems.
Before jumping into the details we shall describe the economic "tools" to do these calculations, i.e. interest rate, annuities, treatment of inflation, etc. The following symbols will be used:

- **A**: annuity ($/year)
- **B**: benefits ($/year)
- **C**: costs ($/year)
- **c**: specific costs ($/liter or $/kWh)
- **E**: annual energy output (kWh/year)
- **e**: escalation rate
- **F**: annual fuel consumption (liter/year)
- **f**: specific fuel consumption (liter/kWh)
- **I**: initial (capital) investment ($)
- **i**: inflation rate
- **L**: technical lifetime (years)
- **N**: loan period (years)
- **P**: payback period (years)
- **R**: interest rate (corrected for inflation)
- **r**: discount rate, interest rate
- **S**: scrap value ($)

The following indices will be used:

- **f**: fuel
- **omr**: operation, maintenance and repair
- **pv**: present value
- **yl**: first year (with y2 : second year, etc)
12.2 Elementary economics

12.2.1 Interest, inflation and present value

Interest plays a central role in our economics and stems from the idea that it is more worthwhile to have a sum of money now than later. So, after borrowing an amount I for a year one has to repay not I but $I(1+r)$ in order to compensate the bank. Lending money to the bank has the inverse effect, the capital grows and after $N$ years the sum, has increased to $I(1+r)^N$.

The interest rates will not be the same in both cases, but that is another matter.

Although the sum of money grows, its value will not grow to the same extent because the general rate of increase of prices, called inflation, will reduce the value of the money accordingly. In the following table both effects are shown.

<table>
<thead>
<tr>
<th>Year</th>
<th>Value of capital in year $N$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>interest only</td>
</tr>
<tr>
<td>0</td>
<td>I</td>
</tr>
<tr>
<td>1</td>
<td>$I(1+r)$</td>
</tr>
<tr>
<td>2</td>
<td>$I(1+r)^2$</td>
</tr>
<tr>
<td>3</td>
<td>$I(1+r)^3$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$N$</td>
<td>$I(1+r)^N$</td>
</tr>
</tbody>
</table>

Fig. 12.1 The influence of interest rate $r$ and inflation rate $i$ on the value of a capital $I$. 
The definition of a "real" interest rate $R$ corrected for the inflation will be clear from table 12.1:

$$1 + R = \frac{1+r}{1+i} \quad (12.1)$$

The concept of the present value can easily be grasped from fig. 12.1. If a capital $I$ grows in $N$ years to $I(1+r)^N$ we can reverse this statement by saying that the value of a sum $I(1+r)^N$ in the year $N$ is equal to $I$ at present. Or, by denoting a capital in the year $N$ by $I(N)$, we can define the present value of $I(N)$ as:

$$\text{present value of } I(N) \text{ is: } \frac{I(N)}{(1+r)^N} \quad (12.2)$$

If inflation has to be included this changes into:

$$\text{present value of } I(N) \text{ is: } I(N) \left(\frac{1+r}{1+i}\right)^N = \frac{I(N)}{(1+i)^N} \quad (12.3)$$

12.2.2 Annual repayments

If one has borrowed a given sum of money from the bank one has to repay the bank (monthly or annually) until the total sum has been repaid. These repayments consist of two parts, the principle (i.e. the payback of the loan itself) and the interest, in a proportion that may vary according to the policies of the bank or the wishes of the client.

We will discuss two widely used types of repayment, the annuity type and the linear repayment type. The annuity is chosen such that the borrowed sum can be repaid in equal annual amounts (annuities) throughout the loan period (fig. 12.2).
In the case of a linear repayment, the same principle must be paid each year (equal to the total sum divided by the number of years of the loan period) plus the interest over the sum left in that year. The result is that the annual repayments decrease linearly with the years (fig. 12.2).

In both cases the sum of the present values of each annual repayment, calculated with the rate of interest used, must yield the initial sum again. This gives us a clue how to calculate the value of the annuity, as illustrated in fig. 12.3.
<table>
<thead>
<tr>
<th>Year</th>
<th>Annuity</th>
<th>Present value of annuities in year 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>A</td>
<td>(\frac{A}{1+r})</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>(\frac{A}{(1+r)^2})</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
<td>(\frac{A}{(1+r)^3})</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>N</td>
<td>A</td>
<td>(\frac{A}{(1+r)^N})</td>
</tr>
</tbody>
</table>

Total present value: \(A \times \frac{1 - (1+r)^{-N}}{r}\)

Fig. 12.3 Illustration of the procedure to find the present values of a series of annuities \(A\).
The sum of the present values is found with the well-known formula for the sum of a series:

\[ a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} = \frac{a(1-x^n)}{1-x} \quad (12.4) \]

In our case \( a = \frac{A}{1+r} \) and \( x = \frac{1}{1+r} \) so we can write the sum of the present values in table 12.3 as:

\[ \text{Total present value is: } A = \frac{A}{1+r} \times \frac{1 - \left(\frac{1}{1+r}\right)^N}{1 - \frac{1}{1+r}} = A * \frac{1-(1+r)^{-N}}{r} \quad (12.5) \]

The factor \( \frac{1-(1+r)^{-N}}{r} \) is called the present worth factor.

We use the inverse of formula (12.5) to determine the annuity to be paid if a capital \( I \) is borrowed for \( N \) years.

\[ \text{annuity: } A = I * \frac{r}{1-(1+r)^{-N}} \quad (12.6) \]

The factor \( \frac{r}{1-(1+r)^{-N}} \) is called the annuity factor, or capital recovery factor.
12.2.3 Costs versus benefits

Apart from the costs to borrow the capital for a windmill or a diesel-powered pump, the following costs are involved: operation, maintenance and repair costs, or OMR-costs, and fuel costs for the diesel engine. These costs tend to escalate annually, at a rate which may differ from the general inflation rate \( i \). With a given escalation rate \( e \) (with an index "f" in case of fuel, and "omr" in case of OMR-costs) and costs that are estimated at \( C \) at the moment of buying the equipment (year 0), this means that in the \( n \)-th year these costs have increased to \( C(1+e)^n \). If the interest rate is \( r \) the present value of these costs of the \( n \)-th year is equal to 
\[
C \frac{1+e^n}{1+r}.
\]

Assuming that these costs pertain during the lifetime \( L \) of the equipment, we can add the present values of all costs, similar to the procedure in table 12.3, to find the total present value of all costs throughout the lifetime \( L \):

\[
C \frac{1+e^L}{1+r} \frac{1 - (1+e)^L}{1 + e} = C \frac{1+e^L}{1+r} \frac{1 - (1+e)^L}{1 + e} \tag{12.7}
\]

For \( r = e \) this reduces to \( C*L \) and for \( r \neq e \) the formula can be rewritten into:

\[
C \left( \frac{1+e}{r-e} \right)^L \left\{ 1 - \frac{1+e^L}{1+r} \right\} \tag{12.8}
\]

It may be clear that a similar formula can be derived for the benefits \( B \) to be expected. In the case of a windmill versus a diesel-powered pump we assume here that the benefits for the windmill are equal to all costs not spent on the diesel. These costs mainly involve fuel costs saved, but could also include a reduction in OMR-costs or even the capital costs if the complete diesel can be replaced.
If we assume that the benefits $B$ increase with the general inflation rate $i$ then the following formula for the total present value of the benefits throughout the lifetime $L$ can be derived:

$$ B \frac{(1+i)^L}{(1+r)^L} \left[ 1 - \frac{(1+i)^L}{(1+r)^L} \right] $$  \hspace{1cm} (12.9)

For $i=r$ this reduces to $B*L$ and for $i \neq r$ it can be rewritten as:

$$ B \left( \frac{1+i}{1+r} \right)^L \left[ 1 - \left( \frac{1+i}{1+r} \right)^L \right] $$ \hspace{1cm} (12.10)

The benefit-cost ratio (B/C ratio) of an investment with a capital $I$ is defined as ($B$ and $C$ taken throughout the lifetime $L$):

$$ \frac{\text{present value of all benefits}}{I + \text{present value of all (non-capital) costs}} $$ \hspace{1cm} (12.11)

The net present value (NPV) at any moment of the same investment is ($B$ and $C$ taken until the given moment):

$$ \text{NPV} = (\text{present value of the benefits}) \text{ minus } (I + \text{present value of the (non-capital) costs}) $$ \hspace{1cm} (12.12)

If we start to calculate the net present value of the costs for example, by starting with the original capital $I$ at year 0 and subsequently adding the present value of the costs in the first year, of the second year and soon, then we are calculating the accumulated present value of the costs. If we perform a similar calculation for the benefits and plot both accumulated present values as a function of the year up to which they are accumulated, we arrive at a graph as shown in fig. 12.4.
Fig. 12.4 Accumulating the present values of costs and benefits and plotting them as a function of the year up to which they are accumulated gives the pay-back period $P$ and the $B/C$ ratio.

The intersection of the cost curve and the benefit curve marks the so-called pay-back period $P$ of the initial capital investment for the given discount rate. It is defined as the number of years needed for the accumulated present value of the benefits to become equal to the accumulated present values of the costs, or in other words the time needed for the Net Present Value to become zero.

In a formula we can write this condition as follows, still assuming constant benefits and constant costs, only affected by $i$ and $e$:

$$B \left( \frac{1+i}{r-i} \right)^{P} \left[ 1 - \left( \frac{1+i}{1+r} \right)^{P} \right] = I + C \left( \frac{1+e}{r-e} \right)^{P} \left[ 1 - \left( \frac{1+e}{1+r} \right)^{P} \right]$$

(12.13)
With \( e=1 \) the following expression for \( P \) can be found:

\[
P = \log \left[ \frac{1 - \frac{I(r-1)}{(B-C)(1+i)}}{\log \frac{1+i}{1+r}} \right]
\]  

(12.14)

If the rate of discount changes, the pay-back period will also change. Higher discount rates result in longer pay-back periods. This means that there must be one discount rate for which the pay-back period becomes equal to the lifetime \( L \). This discount rate is called the **internal rate of return** \( r_i \) and is defined as the discount rate for which the accumulated present value of all costs is equal to the accumulated present value of all benefits.

In a formula, assuming constant values of \( B \) and \( C \), we can write this as:

\[
B \frac{1+i}{r_i-1} \left[ 1 - \left( \frac{1+i}{1+r_i} \right)^L \right] = I + C \frac{1+e}{r_i-e} \left[ 1 - \left( \frac{1+e}{1+r_i} \right)^L \right]
\]  

(12.15)

The value of \( r_i \) must be found via trial and error or numerically.

In the case of windmills usually a scrap value \( S \) remains after the lifetime of the windmill. If the scrap value is estimated to be \( S \) in year 0 then it will be inflated to \( S (1+i)^L \). Its present \( L \) value, i.e. \( S \frac{1+i}{1+r} \) has to be added to the benefits of the investment.
12.3 Costs of a water pumping windmill

As an example we shall calculate the costs of a water pumping windmill. The example is given to demonstrate the calculation procedure and the results cannot be seen as the universal value of windmill costs. In each case the assumptions will have to be changed. Our assumptions are the following:

rotor diameter : \( D = 5 \text{ m} \)
lifeftime : \( L = 10 \text{ years} \)
expected output : \( E = 0.1 \frac{\pi}{4} D^2 \sqrt{V^3 \times 0.76} \text{(kWh/year)} \quad \text{(see formula 2.4)} \)
utilization factor of water output : 60%
investment : \( I = \$1,000 \)
scrap value : \( S = \$100 \)
OMR costs : \( C_{omr} = \$25 \text{ per year} \)
discount rate : \( r = 0.15 \)
inflation rate : \( i = 0.10 \)
loan period : \( N = 10 \text{ years} \)

We assume that the OMR costs escalate with the general inflation rate \( i \).
The present value of the costs is given by formula (12.8) and together with the investment and the scrap value it can be written as:

\[
I + C_{omr} \left[ \frac{1+i}{r-i} \right] \left\{ 1 - \frac{1+i}{1+r} \right\} - S \frac{1+i}{1+r}^L
\]

With the values given above this becomes:

\[
\$1,000 + \$197.4 - \$64.1 = \$1,133.3
\]

The amount of useful energy produced to lift water (here expressed as net hydraulic kWh) depends on the average wind speed. With the expected output formula given, multiplied by 0.6 to take the utilization factor into account, the useful energy of this windmill becomes (\( V \) is the local average wind speed):
\[ E = 10.32 \cdot V^3 \text{ kWh/year} \]

or \( V = 2 \text{ m/s} + E = 83 \text{ kWh/year (hydraulic)} \)

\( V = 3 \text{ m/s} + E = 279 \text{ kWh/year (hydraulic)} \)

\( V = 4 \text{ m/s} + E = 660 \text{ kWh/year (hydraulic)} \)

\( V = 5 \text{ m/s} + E = 1290 \text{ kWh/year (hydraulic)} \)

Dividing the present value of all costs, calculated to be $1,133.3, by the total energy produced during the lifetime of the machine gives us the present energy costs:

\[ V = 2 \text{ m/s} + c_{pv} = $1.37 \text{ per kWh (hydraulic)} \]

\[ V = 3 \text{ m/s} + c_{pv} = $0.41 \text{ per kWh (hydraulic)} \]

\[ V = 4 \text{ m/s} + c_{pv} = $0.17 \text{ per kWh (hydraulic)} \]

\[ V = 5 \text{ m/s} + c_{pv} = $0.09 \text{ per kWh (hydraulic)} \]

It is clear that the local average wind speed has a tremendous effect on the energy costs. The effect of changes in the other parameters, such as interest rate, investment etc, must be calculated by repeating the above procedure and changing only the parameter concerned (assuming tacitly that they are independent). This procedure is called a sensitivity analysis and the results of all these calculations can be shown in one diagram, the so-called spider diagram. In fig. 12.5 such a spider diagram is shown with reference costs equal to $0.41 per kWh, i.e. for \( V = 3 \text{ m/s} \) in our example. We see that the costs are most sensitive for the changes in the investment \( I \) and changes in the average wind speed \( V \), as we would expect. Doubling \( I \) results in nearly 1.9 times higher costs while doubling \( V \) reduces the energy costs to 0.125 of the reference value.

So far we only discussed present values of costs, being a true measure of the life-cycle costs of the windmill. As mentioned in 12.1, the potential buyers tend to compare annual costs and usually only first-year costs. For comparison we shall calculate the annual
costs in the first year of the example given above. The annuity \( A \) can be calculated with formula (12.6) and the result is, with \( I = \$ 1,000, \ r = 0.15 \) and \( N = 10 \) year:

\[
A = \$ 199 \text{ per year.}
\]

The annual OMR costs have to added, \( C_{\text{omr}} = \$ 25/\text{year}, \) increasing with \( i = 0.10 \) each year. After the first year the OMR costs are \( \$ 25 \cdot (1+0.1) = \$ 27.50. \) So the farmer has to pay \( \$ 199 + \$ 27.50 = \$ 226.50 \) after the first year, for a net output of 279 kWh (hydraulic) in our reference situation with \( V = 3 \) m/s. The result is that his energy costs are:

First-year energy costs: \( c_{y1} = \$ 0.81/\text{kWh (hydraulic)} \)

These first-year costs also strongly depend on investment level, average wind speed, etc. and a similar sensitivity analysis as for the present value costs above can be carried out.

This is shown in the spider diagram of fig. 12.6. We see that the effect of changes in \( I \) and \( V \) are about the same if compared with fig. 12.5, but that the influence of the interest rate \( r \) differs dramatically. The reason is that the present value of the costs is dominated by the investment \( I \), which is unaffected by \( r \), and is only slightly influenced by the OMR costs. Note that higher interest rates result in a decrease of the present value of the OMR costs. The first-year costs, however, are dominated by the annuity \( A \) of the investment and the annuity strongly depends upon the interest rate. The first-year costs obviously increase with increasing \( r \).

If the loan had to be repaid in linear repayments, the first-year costs would have been slightly higher: \( \$ 1,000/10 + \$ 1,000 \cdot 0.15 = \$ 250. \) Adding the OMR costs after one year, \( \$ 27.50, \) the total costs become \( \$ 277.50. \) With a net output of 279 kWh/year (in a location with \( V = 3 \) m/s) this results in energy costs of:

First-year energy costs: \( c_{y1} = \$ 0.99/\text{kWh (hydraulic)} \)
12.4 Costs of a diesel powered ladder pump

In Thailand, many low-lift ladder pumps are in use for water lifting, powered by 6-8 HP diesel engines. Their consumption varies widely: 3-8 litres of fuel (new and old engines probably) per day of 10 hours running at a head of 0.7 m with an output of about 50 l/s. This implies that 0.87 to 2.33 litres of fuel is needed to produce 1 kWh (hydraulic). With a calorific value for diesel fuel of 39 MJ/litres (10.8 kWh(th)/litre) the overall efficiency from fuel to water varies between 11% and 4%. Assuming a pump efficiency of 40% [32] this corresponds to engine efficiencies of 27% and 10% respectively.

In our calculations we shall use a specific fuel consumption of 1 litre/kWh (hydraulic), assuming that most of its lifetime the engine can be regarded as an old engine. The effect of higher or lower efficiencies will be analyzed later in a spider diagram. With 10 hours running per day during 60% of the year, the diesel pump in the above typical situation produces 752 kWh (hydraulic) per year. For a comparison with lower windmill outputs we shall have to make assumptions about the reduction in the OMR costs and the possibly extended lifetime of the engine. We shall see, however, that this effect is quite small.

The data used for our calculations are the following:

- investment = $1,000
- technical lifetime = 10 years
- scrap value = $50
- annual OMR costs = $150/year
- fuel cost = $0.4/litre
- escalation of fuel cost = 0.15
- discount rate = 0.15
- inflation rate = 0.10
- specific fuel consumption = 1 litre/kWh (hydraulic)
- annual energy output = 750 kWh (hydraulic)/year
- annual fuel consumption = 750 litre/year
The present value of the costs during the lifetime of the engine can be calculated with formula (12.8). Adding the investment I and subtracting the present value of S gives:

\[ C_{pv} = $1,000 + $1,184 + $3,000 - $32 = $5,216 \]

The total amount of energy produced in 10 years is 7,500 kWh (hydraulic), so the present value of the energy costs per kWh become:

\[ c_{pv} = 0.70/\text{kWh (hydraulic)} \]

We see that the fuel costs dominate the energy costs. In the example they form 58% of the present value of the energy costs. To estimate the effect of a reduction in the annual energy output we shall make the following assumptions:

1. \( E = 400 \text{ kWh (hydraulic)/year} \) : \( C_{omr} = $80/\text{year} \)
   
   \[ L = 15 \text{ years} \]

2. \( E = 200 \text{ kWh (hydraulic)/year} \) : \( C_{omr} = $40/\text{year} \)
   
   \[ L = 20 \text{ years} \]

If all other parameters remain the same the resulting costs are:

1. \( C_{pv} = $1,000 + $856 + $2,400 - $25 = $4,231 \) or \( c_{pv} = 0.71/\text{kWh} \) (hydr)

2. \( C_{pv} = $1,000 + $518 + $1,600 - $20 = $3,098 \) or \( c_{pv} = 0.77/\text{kWh} \) (hydr)

We may conclude that the effect of output reduction on \( c_{pv} \) is probably rather small and this is why we shall neglect this effect in the comparison with the windmill later on.
The effect of changes in one of the parameters without affecting the others can be judged in the spider diagram of fig. 12.7. As expected, the influences of the fuel cost, the fuel escalation rate and the specific fuel consumption are most pronounced.

Similar to the case of the windmill, the present values of the energy costs do not reflect the actual costs which the farmer has to pay to the bank and to his fuel supplier. We can calculate his actual costs after one year with the given data:

\[
\begin{align*}
\text{annuity} & : A = \$199 \\
\text{OMR costs} & : C_{\text{omr}}(1+i) = \$165 \\
\text{fuel costs} & : Fc_f(1+e) = \$345 \\
\text{total costs} : C_{y1} & = \$709
\end{align*}
\]

With an annual output of 750 kWh (hydraulic) per year his energy costs in the first year are:

\[
c_{y1} = 0.95/\text{kWh (hydraulic)}
\]

The spider diagram for the relative changes in these first-year costs is given in fig. 12.8. Note that the effect of the escalation of the fuel costs is rather small, compared with its effect on the long run as shown in the present values of fig. 12.7.
12.5 Comparison of wind and diesel costs

The economics of water pumping windmills with respect to diesel powered pumps can be judged by plotting the accumulated present values of costs and benefits versus time (cf. fig. 12.4). This has been done in fig. 12.9 in which the costs of three Ø 5 m windmills are compared with the benefits of saved fuel in four different wind regimes. The assumptions underlying the graph are as follows:

- Investments windmills: \( I = \$1,000, \$3,000, \$5,000 \)
- OMR costs windmills: \( C_{OMR} = \$25, \$50, \$75 \)
- Average wind speeds: \( V = 2, 3, 4, 5 \text{ m/s} \)
- Annual outputs: \( E = 83, 279, 660, 1290 \text{ kWh(hydr)} \) (utilization factor 60%)
- Interest rate: \( r = 0.15 \)
- Inflation rate: \( i = 0.10 \)
- Fuel costs: \( c_f = \$0.40 \text{ per litre} \)
- Escalation of fuel costs: \( e = 0.15 \)
- Specific fuel consumption: \( f = 1 \text{ litre/kWh} \)

It will be clear from fig. 12.9 that the windmill costs are dominated by the investment \( I \) and much less by the OMR costs. The benefits in terms of fuel saved turn out to be straight lines, because the interest rate \( r \) and the escalation rate \( e \) of the fuel costs are chosen to be the same (see formula 12.8). As expected, the effect of the average wind speed is strong: a $3,000 windmill is paid back in 13 years in a location with \( V = 4 \text{ m/s} \), but the payback time is only about 6 years when \( V = 5 \text{ m/s} \). It will be clear that similar fuel lines can be drawn for other combinations of annual output \( E \), specific fuel consumption \( f \), and fuel costs \( c_f \) (assuming that \( r = e \)). In fact the slope of the benefit line is given by their product:

\[
\text{slope of benefit line (if } r = e \text{): } E \times f \times c_f \text{ ($/year)} \text{ (12.16)}
\]
It is instructive to show the actual annual costs of both systems, to demonstrate how their costs change from year to year. This is shown in fig. 12.10 for the assumptions mentioned above, including a loan period of 10 years. We can see that the actual costs of the $3,000 windmill balance the actual benefits of saved fuel already after about six years (in a location with $V = 4 \text{ m/s}$). We have to wait until year 13, however, until the total present value of both costs do balance, as we have seen in fig. 12.9. In other words, another 7 years, with fuel savings increasingly higher than the windmill costs, are needed to balance the first 6 years with much lower fuel savings.

NOTE: we must stress again that these figures are given as an example, they should not be seen as universal truths.
Fig. 12.5 The relative change in the energy costs of a given water pumping windmill based upon a reference cost of 0.41 $/kWh(hydraulic).
Fig. 12.6 The relative first-year energy costs of a water pumping windmill based upon a reference cost of 0.81 $/kWh, calculated with annuity repayment of the loan.
Fig. 12.7 The relative present value of the energy costs of a diesel-powered pump (described in section 12.4) as a function of the relative changes in the parameters involved. The reference costs are $0.70/kWh(hydraulic).
Fig. 12.8 The relative first-year energy costs of a diesel-powered pump, described in section 12.4, as a function of the relative changes in the parameters involved. The reference first year costs are $0.95/kWh(hydraulic), calculated with annuity repayment of the loan.
Fig. 12.9 The accumulated present values of the costs of three windmills compared with the saved fuel costs (= benefits) of these windmills in four different wind regimes (rotor diameter 5 m, utilization efficiency 60%, other assumptions see section 12.5).
Fig. 12.10 The actual annual costs of three windmills (Ø 5 m) compared with the actual annual costs of the fuel saved in four different wind regimes (assumptions, see section 12.5, and calculations based upon annuity repayment of the loan.
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14. **QUESTIONS**

14.1 **Questions introductory course**

1.1 There are many types of windmills. Give a few characteristics to make a distinction between all these types.

2.1 Give the value of the maximum $C_p$ of an ideal windmill.

2.2 What is the effect of air density upon the output of a windmill?

2.3 When I double the windspeed how much smaller can my wind rotor become in order to arrive at the same output power?

2.4 Estimate the output of a water pumping windmill of $\varnothing$ 5 m pumping at a head of 15 m in a wind regime with $\bar{V} = 4$ m/s.

2.5 If the roughness factor of a given terrain is 0.25 m and the speed at 10 m height is 5 m/s, what is the wind speed at 6 m?

2.6 What is the minimum distance between two windmills in order that the wake of the first is hardly affecting the second?

3.1 Basically two types of manipulations with wind data are possible. Which ones?

3.2 What is the use of a histogram showing the daily wind pattern?

3.3 Calculate the average wind speed of the following frequency distribution:

<table>
<thead>
<tr>
<th>0-1 m/s</th>
<th>10-11 m/s</th>
<th>20-20 m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>285</td>
<td>297</td>
<td></td>
</tr>
<tr>
<td>733</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>945</td>
<td>113</td>
<td></td>
</tr>
<tr>
<td>1088</td>
<td>106</td>
<td></td>
</tr>
<tr>
<td>1193</td>
<td>43</td>
<td></td>
</tr>
<tr>
<td>1127</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>891</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>722</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>556</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>377</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>&gt; 20</td>
<td>26</td>
<td></td>
</tr>
</tbody>
</table>
3.4 What is a velocity duration curve and how to make use of it?
4.1 Why is the maximum torque and the maximum power of a wind rotor not reached at the same rotational speed? And which one is reached at the lower speed?
4.2 What is the difference between the design tip speed ratio and the maximum tip speed ratio?
4.3 From a wind rotor in a wind speed of 8 m/s it is given that $C_p = 0.3$, $\lambda = 6$ and $P = 1000$ W. What is the diameter of the rotor and at which speed does it rotate? What is the torque delivered by this rotor?
4.4 Describe the procedure to find the minimum $C_D/C_L$ ratio of an air foil.
4.5 What is maximum power coefficient that can be attained by a four-bladed rotor equipped with air foils with a minimum $C_D/C_L = 0.05$? At which tip speed ratio?
4.6 Design a blade of a four-bladed wind rotor with the following data: $D = 6.5$ m, $\lambda_0 = 7$, $C_{L_0} = 0.8$, $\alpha_0 = 4^\circ$. Assume that the lift coefficient is constant along the blade.
5.1 What is the main difference between a piston pump and a centrifugal pump?
5.2 Which types of pumps are suitable for high-head low-flow applications? And which pumps for low head and high flow.
5.3 What is the function of air chambers?
5.4 If a piston pump with a diameter of 0.15 m and a stroke of 0.1 m is pumping water over a head of 10 m, then determine the starting torque and the average torque of the pump.
5.5 Assume that the pump of question 5.4 produces 1.5 litre/second at a speed of 1 rev. per second and that it requires an input torque of 3.5 kgm. Calculate the mechanical and volumetric efficiencies of the pump.
6.1 If the pump of question 5.4 is coupled to a rotor with a diameter of 5 m and a design tip speed ratio of 2 then determine the design wind speed of the machine if the maximum overall efficiency is 15%.
6.2 Determine the starting wind speed of the windmill of question 6.1 (hint: use formula 8.5).
7.1 Why is an asynchronous generator asynchronous?

7.2 What is the advantage of an asynchronous generator compared to a synchronous generator for wind energy applications, if both are to be coupled directly to the electricity grid?

7.3 Give an advantage and a disadvantage of a DC commutator machine for wind energy use.

8.1 A generator with the following characteristics has to be coupled to a wind rotor:

\[ P = 15 \text{ kW} \]
\[ n_{\text{in}} = 500 \text{ r.p.m.} \]
\[ n_{\text{r}} = 1500 \text{ r.p.m.} \]
\[ Q_{\text{start}} = 6 \text{ Nm} \]
\[ n(n_{\text{r}}) = 0.85 \]
\[ P_{\text{mech}}(n_{\text{in}}) = 1400 \text{ W} \]

The relevant other data are:

\[ n_{\text{tr}} = 0.9 \]
\[ V_{\text{in}} = 4.5 \text{ m/s} \]
\[ C = 0.35 \]
\[ V_{\text{r}} = 12 \text{ m/s} \]

Find the diameter and the design tip speed ratio of a suitable wind rotor, the transmission ratio of the gearbox and the starting wind speed of the machine.

9.1 Describe a graphical method to calculate the output of a windmill with a given P(V) curve in a given wind regime. Show in a drawing the effect of changing the cut-in speed of the windmill on its output.

9.2 Calculate the output of a \( \phi \) 5 m windmill in the wind regime of Hambantota (section 9.1.2), pumping at a total head of 5 m. Assume that the windmill has a cut-in wind speed of 4 m/s, a rated speed of 10 m/s and a cut-out speed of 14 m/s. The output curve between \( V_{\text{in}} \) and \( V_{\text{r}} \) is linear and the \[ C_{\text{Pmax}} = 0.2 \]

9.3 Estimate the output in the situation of question 9.2 by means of the dimensionless output graphs and give your comments.

10. p.m.

11.1 Which are the two main functions of a safety system?

11.2 Why are there so many different types of safety systems?

11.3 What is the difference between an inclined hinged vane safety system and an eclipsing type?
14.2 Questions advanced course

3.1 Derive the probability density function from \( F(V) = 1 - \exp(-(V/c)^k) \).
3.2 Derive the expression for the average wind speed (3.10).
3.3 Derive \( f(V) \) (3.12) from \( F(V) \) in (3.11).
3.4 What is the power density in a place with a Weibull shape factor of 2.5 and an average wind speed of 4 m/s?
3.5 Derive the expression for the standard deviation of a Weibull distribution.

4.1 What is the difference between the momentum theory and the blade element theory in the aerodynamic analysis of rotors?
4.2 Derive an expression for \( dQ \) with the blade element theory containing "a" and not "a'".
4.3 Find a relation between \( \phi, \lambda_r, a, a' \) different from (4.63).
4.4 Estimate the optimum tip speed ratio of a rotor with four blades, chord 0.2 m, diameter 6.5 m, blade setting angle 4°, assuming a NACA 4412 profile.

5.1 An ideal piston pump with a stroke of 0.25 m is operating at a head of 4 m (suction head), and a suction pipe length of 10 m. Determine the maximum speed before cavitation occurs, if no air chamber is present. What will be the effect of installing a suction air chamber with an effective column length of 1 m?
5.2 If an ideal piston pump with a stroke of 0.25 m operates at a speed of 2 rev/sec, without air chambers, then determine the position angle at which the piston valve will open during the upward stroke. At which angle does the foot valve close? What is the volumetric efficiency of the pump at this speed? And at which speed does the pump become an impulse pump, such that it would even work without a foot valve?
5.3 What is the effect of using heavier valves in piston pumps? And what is the use of reducing the gap width of a valve in a piston pump?
5.4 Advise the volume of a pressure air chamber with the following data given: pressure head 15 m, pipe diameter 0.1 m, pipe length 20 m and a minimum pump speed of 0.5 rev/sec.
6.1 Derive an expression for the power output of a water pumping windmill assuming that the $C_Q^{-\lambda}$ characteristic of the rotor is linear and that the torque characteristic of the pump is quadratic: $Q_p = Q_d (1 + a \Omega^2)$.

6.2 Find an expression for the power coefficient of the rotor of question 6.1.

6.3 Calculate the (static) starting wind speed of a $\varnothing$ 5 m water pumping windmill, with a measured starting torque of 184 Nm at 5 m/s, if the piston pump has a stroke of 0.1 m, a diameter of 0.2 m, operates at a head of 5 m and has a pump rod with a total weight of 30 kg.

6.4 A piston pump of 10 cm diameter, stroke 6 cm, operating at a head of 5 m, is equipped with a leakhole of $\varnothing$ 3 mm, length 10 mm. What is its minimum rotational speed to produce water? At which position angle does the delivery start when the pump operates at 2 rev/sec? What is the average torque at that speed and what is the discharge? Calculate also the volumetric efficiency of the pump.

7. p.m.

8.1 Determine the design speed of a wind turbine with $P(V) = a V^{1.5} - b$.

8.2 Derive an expression for $C_p\Omega(V)$ of this turbine.

9.1 Derive an expression for the annual energy output of a wind turbine with a linear output, as a function of $\Omega$, $V_{in}$, $V_r$, $V_{out}$ and $k$.

9.2 Find the expression for $e_{system}$ of the wind turbine of question 9.1.

9.3 What is the maximum value of $e_{system}$ of an ideal windmill in a $k = 2.5$ regime?

9.4 What is the maximum value of $e_{system}$ of a wind turbine with a linear output characteristic in a $k = 2$ wind regime? And which value of $e_{system}$ can be reached for $x_d = 1$?
Examination introductory course (AIT, June 1981)

A.1 Generally windmills are classified as being either horizontal axis or vertical axis types. Are there any other types and if so, give one example.

A.2 The maximum power coefficient of an ideal wind rotor is usually related to the undisturbed flow of air reaching the swept area of the rotor and has a value of 16/27. If one wishes to relate the power coefficient to the real mass flow through the rotor instead, what is its maximum value in that case?

A.3 If the owner of a Ø 5 m wind rotor wants to change his rotor in order to arrive at the same output in a 20% lower wind speed, which diameter rotor does he need?

A.4 Estimate the annual output of a Ø 3 m diameter water pumping windmill, operating in a wind regime with an annual average wind speed of 4.5 m/s, if the windmill pumps at 12 m head.

A.5 What is the minimum CD/CL ratio needed to find a maximum power coefficient of at least 0.4 for a four-bladed wind rotor with λd = 5?

A.6 Give three methods to decrease the design wind speed of a water pumping windmill by 20%.

B.1 Estimate the maximum annual output of a Ø 5 m water pumping windmill, operating in the Hambantota (section 9.1.2, assume k = 2). The following data are given: head: 10 m, V / \sqrt{r} = 2, maximum overall efficiency is 0.2, λd = 2 and the output curve is assumed to be linear. Determine also the necessary design wind speed and the necessary stroke for a single-acting piston pump with a diameter of 0.2 m.

B.2 A square flat plate in a flow of air experiences a force in a direction normal (perpendicular) to the plate, irrespective of the angle of attack. The normal force coefficient C_N is a linear function of α, with C_N = 0 at α = 0 and C_N = 1.6 at α = 40°. If the square flat vane of a windmill, dimensions 2 x 2 m, experiences a wind speed of 5 m/s, calculate the force that drives the vane back to its equilibrium position if at a given moment its deviation angle is 30°.
B.3 Calculate the design velocity of a wind turbine with an output characteristic given by:
\[ P(V) = \text{constant} \times (V^2 - V_{\text{in}}^2) \text{ for } V > V_{\text{in}}. \]

B.4 Calculate the starting wind speed of a wind turbine with the following data:
- Rotor: \( \lambda_d = 6 \)
- Transmission: \( i = 8 \)
- \( D = 10 \text{ m} \)
- Generator: \( Q_{\text{start}} = 5 \text{ Nm} \)

C.1 Why is the coupling of a generator to a wind rotor so much more complicated than coupling a pump to a wind rotor? Indicate the aim of the coupling procedure in one sentence.

C.2 Why is it necessary to possess the \( C_L - \alpha \) curve of a profile if one wishes to design a blade with a constant chord for a wind rotor?

C.3 A water pumping windmill usually has a low efficiency at high wind speeds. Why is this and which solution do you propose to solve this drawback?

14.4 Examination advanced course (AIT, July 1981)

1. If a piston pump with a stroke of 0.2 m must be operated at speeds up to 2 rev/sec, which type of damage might possibly occur? Give your solution to prevent the danger.

2. Give an expression for the thrust coefficient \( C_T \) and the power coefficient \( C_p \) of an ideal wind rotor as a function of the axial induction factor. Determine their value at maximum power extraction and draw their graphs.

3. Estimate the dimensionless energy output \( e_{\text{system}} \) of a water pumping windmill (constant torque load) in a wind regime with a Weibull factor \( k = 2 \) and an average wind speed of 5 m/s, if the windmill has a design speed of 5 m/s, a rated speed of 9 m/s and an infinite cut-out speed.
4. The Weibull velocity distribution function $f(V)$ has a maximum. Derive an expression for the wind speed $V_m$ at which this maximum occurs and estimate the value of the Weibull shape factor $k$ for which $V_m = \bar{V}$, in which $\bar{V}$ is the average wind speed of the velocity distribution.

5. Calculate the leak flow of a piston of $\phi 0.1 \text{ m}$ with a leakhole of $\phi 3 \text{ mm}$ and length 6 mm, operating at a head of 10 m. If the pump starts pumping water at a speed of 0.25 rev/sec, what is its stroke?

6. Estimate the minimum volume of an air chamber for a piston pump with a suction line of 100 m length, pipe diameter 5 cm, total suction head 4 m, when the minimum observed pump speed is 0.15 rev/sec.

7. Give a mathematical expression to approximate the water output of a water pumping windmill as a function of the wind speed $V$ if the output at the design speed of 3 m/s is measured to be 1 litre/sec, while at 6 m/s the output has increased to 2.9 litre/sec.

8. Describe the effect of fitting springs to the valves of a reciprocating piston pump, in such a way that the springs tend to push the valves back to their closed position.

9. A hinged vane safety system is meant to keep the rotor of a windmill into the wind at low wind speeds and to turn it out of the wind at high wind speeds. If one could suddenly decrease the angle $\varepsilon$ between the hinge axis and the vertical axis to a value of zero, what would be the effect? And what would be the behaviour of the system if the angle $\varepsilon$ would be increased to twice its original value?

10. Calculate the payback time of a windmill costing $2000, with annual operation and maintenance costs of $50/year, when the rate of interest is 15% and the general inflation rate (also for OMR costs) is 10%.

It is assumed that the benefits of the windmill consist only of the saving of a yearly amount of 600 litre of fuel. The fuel costs are $0.50 per litre of fuel, increasing at a rate of 10% per year.
APPENDIX A

Air density

The density of a gas is proportional to the pressure and inversely proportional to the absolute temperature according to the law of Boyle-Gay-Lussac:

\[ \rho = \frac{M \cdot p}{R \cdot T} \quad \text{(A.1)} \]

in which:
- \( \rho \): density \((\text{kg/m}^3)\)
- \( M \): molecular weight \((\text{kg/mol})\)
- \( p \): pressure \((\text{N/m}^2)\)
- \( R \): universal gas constant \(= 8.31434 \text{ J/mol K} \)
- \( T \): absolute temperature \((\text{K})\)

Knowing that a standard mol of air weighs 28.966 gram \([33]\), or \( M = 0.028966 \text{ kg/mol} \), we can calculate the density of (dry) air at various temperatures and pressures. For example: the density of dry air at a temperature of 30 °C \(= (30 + 273.16) \text{ K} = 303.16 \text{ K} \) and at a standard pressure of 1 atm \(= 1.01325 \times 10^5 \text{ N/m}^2 \) is equal to:

\[ \rho = \frac{0.028966 \times 101325}{8.31434 \times 303.16} = 1.164 \text{ kg/m}^3 \]
Some typical values are given in fig. A.1.

<table>
<thead>
<tr>
<th>Temperature °C</th>
<th>Density of dry air kg/m³</th>
<th>Density of saturated air kg/m³</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20</td>
<td>1.394</td>
<td>1.394</td>
</tr>
<tr>
<td>-15</td>
<td>1.367</td>
<td>1.367</td>
</tr>
<tr>
<td>-10</td>
<td>1.341</td>
<td>1.340</td>
</tr>
<tr>
<td>-5</td>
<td>1.316</td>
<td>1.314</td>
</tr>
<tr>
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<td>1.289</td>
</tr>
<tr>
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<td>1.269</td>
<td>1.265</td>
</tr>
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<tr>
<td>15</td>
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<tr>
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<td>1.146</td>
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<tr>
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<td>1.127</td>
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</tr>
<tr>
<td>45</td>
<td>1.109</td>
<td>1.070</td>
</tr>
<tr>
<td>50</td>
<td>1.092</td>
<td>1.043</td>
</tr>
</tbody>
</table>

Fig. A.1 The densities of dry and saturated air at standard atmospheric pressure at sealevel, i.e. $1.01325 \times 10^5$ N/m².

With increasing height above sea level, the pressure decreases and the temperature also decreases. When the temperature and pressure are known, the density can be calculated with the above formula A.1. If these data are not available, one can use the standard atmospheric conditions, described as follows:
$p_0$ at sea level : $1.01325 \times 10^5$

$T_0$ at sea level : 288.16 K (15 °C)

$T$ at height $z$ : $T_0 - \alpha \times z$

with $\alpha = 0.0065 \text{ K/m}$

This model with a linearly decreasing temperature is reasonably accurate up to a height of 10,000 m i.e. in the troposphere.

Using the basic equation for hydrostatic pressure variation [34]:

$$\frac{dp}{dz} = -\rho g = -\frac{M g}{R(T_0 - \alpha z)} \quad (A.2)$$

one can find by integration:

$$p = p_0 \left[ 1 - \frac{\alpha z}{T_0} \right]^\frac{M g}{\alpha R} \quad (A.3)$$

and so

$$p = \frac{M}{R} \left( 1 - \frac{\alpha z}{T_0} \right)^\frac{M g}{\alpha R} \quad (A.4)$$

This formula is tabulated in table A.2 for dry air.
<table>
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Fig. A.2 The density of dry air at different altitudes under standard atmospheric conditions.

A third, minor, effect on the air density is the presence of water vapour which decreases the density. The decrease depends upon the ratio of vapour pressure e and atmospheric pressure p:

\[ \rho_{\text{wet}} = \rho_{\text{dry}} \left[ 1 - 0.3783 \times \frac{e}{p} \right] \]  

(A.5)

The values of \( \rho \) at different temperatures can be found in [33]. As can be seen in fig. A.1 the effect of the water vapour is rather small, even for completely saturated air.
APPENDIX B

Gyroscopic effects

Simultaneous yawing of the head around the z-axis and turning of the rotor around the x-axis cause acceleration forces and moments, known as gyroscopic effects.

In order to determine the magnitude of these forces and moments we consider the co-ordinate system as shown in fig. B.1.

Fig. B.1 Co-ordinate system for calculation of gyroscopic effect.
The co-ordinates of a mass element \( dm \) of the rotor can be written as:

\[
\begin{align*}
x &= -f \cos \theta - r \sin \theta \sin \phi \\
y &= -f \sin \theta + r \sin \theta \cos \phi \\
z &= -r \cos \theta
\end{align*}
\]  

(B.1)

The force exerted by a mass element \( dm \) upon the rest of the blade is:

\[
\begin{align*}
dF_x &= -x \, dm \\
dF_y &= -y \, dm \\
dF_z &= -z \, dm
\end{align*}
\]  

(B.2)

The values of \( x, y, \) and \( z \) can be determined by differentiating equation (B.1) twice:

\[
\begin{align*}
x &= -f \cos \theta - r \sin \theta \sin \phi \\
\dot{x} &= +f \sin \theta \sin \phi - r \cos \theta \sin \phi \sin \phi \\
\ddot{x} &= +f \cos \theta \left( \dot{\phi}^2 - 2 \, r \cos \theta \sin \phi \right) + f \sin \theta \sin \phi \dot{\phi} \sin \phi - r \cos \theta \sin \phi \cos \phi + r \sin \theta \sin \phi \left( \dot{\phi}^2 - r \sin \theta \cos \phi \right)
\end{align*}
\]  

(B.3)

* Note: Notation \( \dot{x} \) resp. \( x \) means \( \frac{dx}{dt} \) resp. \( \frac{d^2x}{dt^2} \)
\[ y = - f \sin \theta + r \sin \theta \cos \theta \]

\[ \dot{y} = - f \cos \theta \dot{\theta} + r \cos \theta \cos \theta \theta - r \sin \theta \sin \theta \theta \]

\[ \ddot{y} = + f \sin \theta \theta^2 - f \cos \theta \theta^2 - r \sin \theta \cos \theta \theta^2 - r \cos \theta \sin \theta \theta^2 - \]

\[ - r \cos \theta \sin \theta \theta \theta - r \cos \theta \cos \theta \theta \theta - r \cos \theta \sin \theta \theta \theta \]

\[ y = y \]

\[ \theta = \theta \]

\[ \gamma = \gamma \]

\[ \phi = \phi \]

\[ (B.4) \]

\[ \dot{y} = - r \sin \theta \theta^2 - r \sin \theta \theta^2 = r \sin \theta \Omega^2_x - r \sin \theta \Omega^2_z \]

\[ z = - r \cos \theta \]

\[ \dot{z} = r \sin \theta \theta \]

\[ \ddot{z} = r \cos \theta \theta^2 + r \sin \theta \theta \]

\[ \theta = 0, \theta = \Omega_x \]

\[ \gamma = \gamma \]

\[ \phi = \phi \]

\[ (B.5) \]
Substituting (B.3), (B.4) and (B.5) in (B.2) gives:

\[ dF = - f \Omega^2 \frac{\partial}{\partial z} dm + 2 r \cos \theta \Omega \frac{\partial}{\partial z} dm \]

\[ dF = r \sin \theta \Omega^2 \frac{\partial}{\partial x} dm + r \sin \theta \Omega^2 \frac{\partial}{\partial z} dm \]

\[ dF = - r \cos \theta \Omega^2 \frac{\partial}{\partial x} \\
\]

We shall now express \( dF \), \( dF \) and \( dF \) in terms of axial, tangential and radial forces, using the co-ordinate system from fig. B.2.

Fig. B.2. Co-ordinate system for a rotor blade.
\[
dF_{\text{bia}} = df_x
\]

\[
dF_{\text{bit}} = dF_y \cos \theta + dF_z \sin \theta
\]

\[
dF_{\text{bir}} = dF_y \sin \theta - dF_z \cos \theta
\]

Substituting (B.6) in (B.7) gives:

\[
dF_{\text{bia}} = -f \frac{\Omega^2}{z} \frac{dm}{x} + 2r \cos \theta \frac{\Omega}{z} \frac{dm}{x} + \frac{r \cos \theta}{z} \frac{dm}{x}
\]

\[
dF_{\text{bit}} = r \sin \theta \cos \theta \frac{\Omega^2}{x} \frac{dm}{x} + r \sin \theta \cos \theta \frac{\Omega^2}{z} \frac{dm}{z} - r \cos \theta \sin \theta \frac{\Omega^2}{x} \frac{dm}{x}
\]

\[
dF_{\text{bir}} = r \sin^2 \theta \frac{\Omega^2}{x} \frac{dm}{x} + r \sin^2 \theta \frac{\Omega^2}{z} \frac{dm}{z} + r \cos^2 \theta \frac{\Omega^2}{x} \frac{dm}{x}
\]

These expressions reduce to:

\[
dF_{\text{bia}} = -f \frac{\Omega^2}{z} \frac{dm}{x} + 2r \cos \theta \frac{\Omega}{z} \frac{dm}{x}
\]

\[
dF_{\text{bit}} = r \sin \theta \cos \theta \frac{\Omega^2}{z} \frac{dm}{z}
\]

\[
dF_{\text{bir}} = r \frac{\Omega^2}{x} \frac{dm}{x} + r \sin^2 \theta \frac{\Omega^2}{z} \frac{dm}{z}
\]

The total forces \( F_{\text{a}}, F_{\text{t}} \) and \( F_{\text{r}} \) on a rotor blade can be calculated by integrating the equations (B.8). For one blade: \( f = 0 \).

\[
F_{\text{bia}} = \frac{\int_0^R 2r \cos \theta \frac{\Omega}{z} \frac{dm}{x} = 2 \cos \theta \frac{\Omega}{z} \frac{dm}{x}}{o}
\]

\[
F_{\text{bit}} = \frac{\int_0^R r \sin \theta \cos \theta \frac{\Omega^2}{z} \frac{dm}{z} = \sin \theta \cos \theta \frac{\Omega^2}{z} \frac{dm}{z}}{o}
\]

\[
F_{\text{bir}} = \frac{\int_0^R r \frac{\Omega^2}{x} \frac{dm}{x} + \int_0^R r \sin^2 \theta \frac{\Omega^2}{z} \frac{dm}{z} = \frac{\Omega^2}{x} \frac{dm}{x} + \frac{\Omega^2}{z} \frac{dm}{z}}{o}
\]
When calculating the bending moments acting on a rotor blade at the hub, only $dF_{bit}$ and $dF_{bia}$ cause moments. $dF_{bit}$ causes a bending moment $M_{bia}$ on the axial axis:

$$dM_{bia} = dF_t \times r = r^2 \sin \theta \cos \theta \Omega^2 \, dm$$  \hspace{1cm} (B.10)

Hence:

$$M_{bia} = \int_{o}^{R} r^2 \sin \theta \cos \theta \Omega^2 \, dm = \sin \theta \cos \theta \Omega^2 \int_{o}^{R} r^2 \, dm$$  \hspace{1cm} (B.11)

$dF_{bia}$ causes a bending moment on the tangential axis

$$dM_{bit} = dF_a \times r = 2r \cos \theta \Omega \, dm \, r$$  \hspace{1cm} (B.12)

Hence:

$$M_{bit} = \int_{o}^{R} 2r^2 \cos \theta \Omega \, dm = 2 \cos \theta \Omega \int_{o}^{R} r^2 \, dm$$  \hspace{1cm} (B.13)

Since $\int dm = J_b$, the first mass moment of inertia of a blade and $\int r^2 \, dm = I_b$, the (second) mass moment of inertia of a blade, the gyroscopic forces and moments for one blade can be written as:

$$F_{bia} = 2 \cos \theta \Omega \Omega J_b$$

$$F_{bit} = \sin \theta \cos \theta \Omega^2 J_b$$  \hspace{1cm} (B.14)

$$F_{bir} = \Omega^2 J + \sin^2 \theta \Omega^2 J_b$$

With these expressions the moments become:

$$M_{bia} = \sin \theta \cos \theta \Omega^2 I_b$$  \hspace{1cm} (B.15)

$$M_{bit} = 2 \cos \theta \Omega \Omega I_b$$
As an example, the gyroscopic forces and moments have been calculated for the WEU I-3.

Data for the WEU I-3 rotor blade are:

- Radius of rotor: $1.5 \text{ m}$
- Mass of a rotor blade: $4.6 \text{ kg}$
- $\Omega_x = 15.7 \text{ rad/s}$
- $\Omega_z = .63 \text{ rad/s}$
- Cross-sectional area rotor spoke $A_s = 192 \text{ mm}^2$
- Moment of inertia $I/c = 1.08 \times 10^3 \text{ mm}^3$

When the mass of a rotor blade is assumed to be uniformly distributed over the blade, $J$ can be calculated:

$$J_b = \int_0^R r \ dm = \rho A \int_0^R \ dr = \rho Ar \frac{r^2}{2} = \frac{mr}{2} = 3.45 \text{ kgm}$$

$$I_b = 3.6 \text{ kgm}^2$$

Hence:

$$F_{bia} = 68 \cos \theta \quad N$$

$$F_{bit} = 1.4 \sin \theta \cos \theta \quad N$$

$$F_{bir} = 850 + 1.4 \sin^2 \theta \quad N$$

$$M_{bia} = 1.4 \sin \theta \cos \theta \quad \text{Nm}$$

$$M_{bit} = 71.2 \cos \theta \quad \text{Nm}$$

$F_{bia}$ and $F_{bit}$ are shearing forces, causing a maximum shearing tension $\tau$.
\[
\tau = \frac{F_{\text{bir}} + F_{\text{bit}}}{A} = 0.35 \text{ N/mm}^2
\]

\(F_{\text{bir}}\) causes a maximum tension \(\sigma\):

\[
\sigma = \frac{F_{\text{bir}}}{A} = 4.4 \text{ N/mm}^2
\]

\(M_{\text{bia}}\) and \(M_{\text{bit}}\) cause a resultant maximum bending stress:

\[
\sigma_b = 65 \text{ N/mm}^2
\]

As can be seen from the calculated stresses, the stresses due to the gyroscopic forces can be neglected with respect to the stresses due to the gyroscopic moments.
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